

# Hamilton-Jacobi equations for optimal control on multidimensional junctions

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December 10, 2014

## Abstract

We consider continuous-state and continuous-time control problems where the admissible trajectories of the system are constrained to remain on a union of half-planes which share a common straight line. This set will be named a junction. We define a notion of constrained viscosity solution of Hamilton-Jacobi equations on the junction and we propose a comparison principle whose proof is based on arguments from the optimal control theory.

**Keywords** Optimal control, junctions, Hamilton-Jacobi equations, viscosity solutions

## 1 Introduction

We are interested in optimal control problems whose trajectories are constrained to remain on a multidimensional junction. We define a *junction* in  $\mathbb{R}^d$ ,  $d \geq 2$ , as a union of half-hyperplanes sharing an affine space of dimension  $d - 2$ , see Figure 1 for a junction in  $\mathbb{R}^3$ . For simplicity, we shall limit ourselves to junctions in  $\mathbb{R}^3$ , although all what follows can be generalized for  $d \geq 3$ . We shall name *interface* the straight line  $\Gamma$  shared by the half-planes.

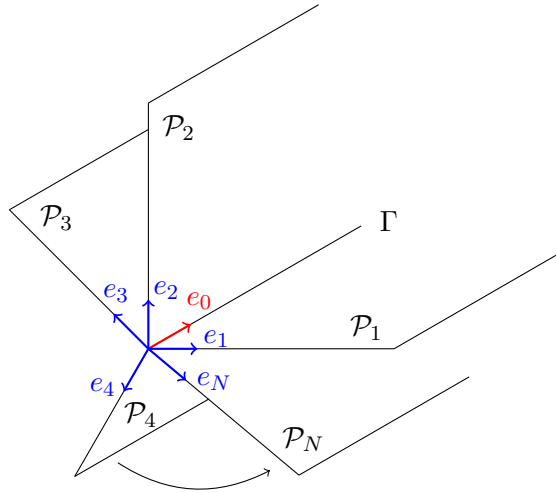


Figure 1: The junction  $\mathcal{S}$  in  $\mathbb{R}^3$

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The junctions in  $\mathbb{R}^d$  are particular *ramified sets*, which we define as closed and connected subsets of  $\mathbb{R}^d$  obtained as the union of embedded manifolds with dimension strictly smaller than  $d$ . Figure 2 below supplies examples of ramified sets.

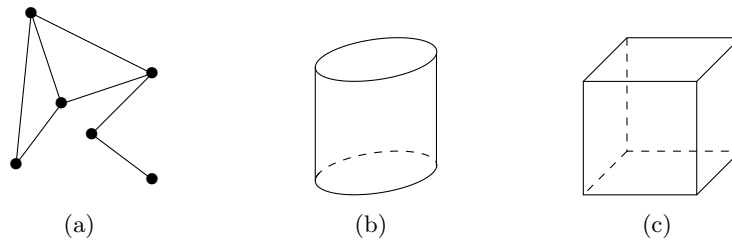


Figure 2: Examples of ramified sets

While there is a wide literature on control problems with state constrained in closures of open sets ([23, 24], [9], [16]), the interest on problems with state constrained in closed sets with empty interior is more recent. The results of Frankowska and Plaskacz [11, 10] do apply to some closed sets with empty interior, but not to ramified sets except in very particular cases.

The case of Hamilton-Jacobi equations on networks, see Figure 2 (a), is now well understood. The first work seems to be the thesis of Schieborn in 2006, [21]. It was focused on eikonal equations. These results were improved later in [22]. The notion of viscosity solutions in these works are restricted to eikonal equations and cannot be used for more general control problems. The first two articles on optimal control problems whose dynamics are constrained to a network were published in 2013: in [1] Achdou, Camilli, Cutrì and Tchou proposed a Hamilton-Jacobi equation and a definition of viscosity solution. Independently, in [15] Imbert, Monneau and Zidani proposed a Hamilton-Jacobi approach to junction problems and traffic flows and gave an equivalent notion of viscosity solution. Both [1] and [15] contain the first comparison and uniqueness results for Hamilton-Jacobi equations on networks, but these results needed rather strong assumptions on the Hamiltonians behaviour. A general comparison result has finally been obtained in the recent paper by Imbert-Monneau [14]. In the latter, the Hamiltonians in the edges are completely independent from each other; the main assumption is that the Hamiltonian in each edge, say  $H_i(x, p)$  for the edge indexed  $i$ , is coercive and bimonotone, i.e. non increasing (resp. non decreasing) for  $p$  smaller (resp. larger) than a given threshold  $p_i^0(x)$ . Of course, convex Hamiltonian coming from optimal control theory are bimonotone. Moreover, in [14], the authors consider more general transmission conditions than in [1, 15], allowing an additional running cost at the junctions. Soon after, Y. Achdou, S. Oudet and N. Tchou, [2], proposed a different proof of a general comparison result in the case of control problems. In [2], the dynamics and running costs be different on each edges, and the Hamiltonians in the edges are a priori completely independent from each other, as in [14]. Whereas the proof of the comparison result in [14] it is only based on arguments from the theory of partial differential equations, the proof in [2] is based on arguments from the theory of control which were first introduced by G. Barles, A. Briani and E. Chasseigne in [6, 7]. In the latter articles, the authors study control problems in  $\mathbb{R}^N$  whose dynamics and running costs may be discontinuous across an hyperplane. The problems studied in [6, 7] and in [2] have the common point that the data are discontinuous across a low dimensional subregion.

There is even less literature on Hamilton-Jacobi equations posed on more general ramified spaces. In their recent article [8], F. Camilli, D. Schieborn and C. Marchi deal with eikonal equations, generalize the special notion of viscosity solutions proposed in [21, 22], and prove existence and uniqueness theorems. The work by Y. Giga, N. Hamamuki and A. Nakayasu, see [12], is devoted to eikonal equations in general metric spaces, and their results apply to ramified spaces. The difficulty with general metric space  $(\chi, d)$  is that the gradient  $Du$  of

a function  $u : \chi \rightarrow \mathbb{R}$  is not well-defined in general. Yet, eikonal equation can be studied since a definition for the modulus of the gradient can be given. More general results in geodesic metric spaces have been recently given by L. Ambrosio and J. Feng, see [3], see also the recent paper of [20], who considered evolutionary Hamilton-Jacobi equations of the form  $u_t + H(x, |Du|) = 0$  in a metric space. For optimal control problems on ramified sets, we can mention the recent article by C. Hermosilla and H. Zidani [13], in which they study infinite horizon problems whose trajectories are constrained to remain in a set with a stratified structure. The authors obtain existence and uniqueness results with weak controllability assumptions, but they assume that the dynamics is continuous.

The present work is a continuation of [2] (which was focused on networks), but since the interface  $\Gamma$  is now a straight line instead of a point, the trajectories that stay on  $\Gamma$  have a richer structure than in [2]. We will have to introduce a tangential Hamiltonian  $H_\Gamma^T$  to take these admissible trajectories into account. Different controllability assumptions can be made

1. strong controllability in a neighborhood of  $\Gamma$
2. a weaker controllability assumption in a neighborhood of  $\Gamma$ , namely normal controllability to  $\Gamma$ , see  $[\tilde{H}_3]$  in § 2.1.2.

As in [2], the proof of the comparison results will be inspired by the arguments contained in [6, 7].

In § 2, we discuss the case when strong controllability is assumed in a neighborhood of  $\Gamma$ . We propose a Bellman equation and a notion of viscosity solutions, prove that the value function is indeed a continuous viscosity solution of this equation, and give a comparison result. In § 3, the same program is carried out when only normal controllability holds in a neighborhood of  $\Gamma$ . As in [7], we first prove that the value function is a discontinuous viscosity solution of the Bellman equation. We then prove a comparison result. The latter implies the continuity of the value function. Finally, in § 4, we extend the results by assuming that in addition to the dynamics and costs related to the hyperplanes, there is a pair of tangential dynamics and tangential running cost defined on  $\Gamma$ .

Although we will not discuss it, the results obtained below can be generalized to ramified sets for which the interfaces are non intersecting manifolds of dimension  $d - 2$ , see for example Figure 2 (b). On the contrary, it is not obvious to apply them to the ramified sets for which interfaces of dimension  $d - 2$  cross each other, see Figure 2 (c). This topic will hopefully be discussed in a forthcoming work.

## 2 First case : full controllability near the interface

### 2.1 Setting of the problem and basic assumptions

#### 2.1.1 The geometry

We are going to study optimal control problems in  $\mathbb{R}^d$ ,  $d = 3$ , with constraints on the state of the system. The state is constrained to lie in the union  $\mathcal{S}$  of  $N$  half-planes,  $N > 1$ . Let  $(e_i)_{i=0,\dots,N}$ , be some respectively distinct unit vectors in  $\mathbb{R}^d$  such that  $e_i \cdot e_0 = 0$  for any  $i \in \{1, \dots, N\}$ , where for  $x, y \in \mathbb{R}^d$ ,  $x \cdot y$  denotes the usual scalar product of the Euclidean space  $\mathbb{R}^d$ . The notation  $|\cdot|$  will be used for the usual Euclidean norm in  $\mathbb{R}^d$ . For  $i = 1, \dots, N$ ,  $\mathcal{P}_i$  is the closed half-plane  $\mathbb{R}e_0 \times \mathbb{R}^+ e_i$ . We denote by  $\Gamma$  the straight line  $\mathbb{R}e_0$ . The half-planes  $\mathcal{P}_i$  are glued at the straight line  $\Gamma$  to form the set  $\mathcal{S}$ , see Figure 1:

$$\mathcal{S} = \bigcup_{i=1}^N \mathcal{P}_i.$$

For any  $x \in \mathcal{S}$ , we denote by  $T_x(\mathcal{S}) \subset \mathbb{R}^d$  the set of the tangent directions to  $\mathcal{S}$ , i.e.  $T_x(\mathcal{S}) = \mathbb{R}e_0 \times \mathbb{R}e_i$ , for any  $x \in \mathcal{P}_i \setminus \Gamma$  and  $T_x(\mathcal{S}) = \cup_{i=1}^N (\mathbb{R}e_0 \times \mathbb{R}e_i)$  for any  $x \in \Gamma$ .

If  $x \in \mathcal{S} \setminus \Gamma$ ,  $\exists! i \in \{1, \dots, N\}$ ,  $\exists! x_0 \in \mathbb{R}$  and  $\exists! x_i \in \mathbb{R}_+ \setminus \{0\}$  such that

$$x = x_0 e_0 + x_i e_i. \quad (2.1)$$

We will use often this decomposition. Sometime, it will be convenient to extend this decomposition to the whole set  $\mathcal{S}$ , by writing  $x = x_0 e_0 + 0 e_i$  for any  $i \in \{1, \dots, N\}$  if  $x \in \Gamma$ . When we will not want to specify in which half-plane  $\mathcal{P}_i$  is  $x$  belonging to  $\mathcal{S}$ , we will use the notation

$$x = x_0 e_0 + x', \quad (2.2)$$

where  $x'$  denotes  $x_i e_i$  if  $x = x_0 e_0 + x_i e_i$ .

The geodesic distance  $d(x, y)$  between two points  $x, y$  of  $\mathcal{S}$  is

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \text{ belong to the same half-plane } \mathcal{P}_i \\ \min_{z \in \Gamma} \{|x - z| + |z - y|\} & \text{if } x, y \text{ belong to different half-planes } \mathcal{P}_i \text{ and } \mathcal{P}_j. \end{cases} \quad (2.3)$$

More classically, if  $x \in \mathbb{R}^d$  and  $C$  is a closed subset of  $\mathbb{R}^d$ ,  $\text{dist}(x, C)$  will denote

$$\text{dist}(x, C) = \inf\{|x - z| : z \in C\},$$

the distance between  $x$  and  $C$ . The notation  $B(\Gamma, r)$  will be used to denote the set  $\{x \in \mathbb{R}^d : \text{dist}(x, \Gamma) < r\}$ .

### 2.1.2 The optimal control problem

We consider infinite-horizon optimal control problems which have different dynamics and running costs in the half-planes. We are going to describe the assumptions on the dynamics and costs in each half-plane  $\mathcal{P}_i$ : the sets of controls are denoted by  $A_i$ , the system is driven by the dynamics  $f_i$  and the running costs are given by  $\ell_i$ . Our main assumptions are as follows

[H0]  $A$  is a metric space (one can take  $A = \mathbb{R}^m$ ). For  $i = 1, \dots, N$ ,  $A_i$  is a non empty compact subset of  $A$  and  $f_i : \mathcal{P}_i \times A_i \rightarrow \mathbb{R} e_0 \times \mathbb{R} e_i$  is a continuous bounded function. The sets  $A_i$  are disjoint. Moreover, there exists  $L_f > 0$  such that for any  $i$ ,  $x, y \in \mathcal{P}_i$  and  $a \in A_i$ ,

$$|f_i(x, a) - f_i(y, a)| \leq L_f |x - y|.$$

We note  $M_f$  the minimal constant such that for any  $x \in \mathcal{S}$ ,  $i \in \{1, \dots, N\}$  and  $a \in A_i$ ,

$$|f_i(x, a)| \leq M_f.$$

We will use also the notation  $F_i(x)$  for the set  $\{f_i(x, a), a \in A_i\}$ .

[H1] For  $i = 1, \dots, N$ , the function  $\ell_i : \mathcal{P}_i \times A_i \rightarrow \mathbb{R}$  is a continuous and bounded function. There is a modulus of continuity  $\omega_\ell$  such that for all  $i \in \{1, \dots, N\}$ ,  $x, y \in \mathcal{P}_i$  and  $a \in A_i$ ,

$$|\ell_i(x, a) - \ell_i(y, a)| \leq \omega_\ell(|x - y|).$$

We denote  $M_\ell$  the minimal constant such that for any  $i \in \{1, \dots, N\}$ ,  $x \in \mathcal{P}_i$  and  $a \in A_i$ ,

$$|\ell_i(x, a)| \leq M_\ell.$$

[H2] For  $i = 1, \dots, N$ ,  $x \in \mathcal{P}_i$ , the non empty and closed set

$$\text{FL}_i(x) = \{(f_i(x, a), \ell_i(x, a)), a \in A_i\}$$

is convex.

[H3] There is a real number  $\delta > 0$  such that for any  $i = 1, \dots, N$  and for all  $x \in \Gamma$ ,

$$B(0, \delta) \cap (\mathbb{R}e_0 \times \mathbb{R}e_i) \subset F_i(x).$$

In § 3 below, we will weaken assumption [H3] and use only the assumption on normal controllability

[H3] There is a real number  $\delta > 0$  such that for any  $i = 1, \dots, N$  and for all  $x \in \Gamma$ ,

$$[-\delta, \delta] \subset \{f_i(x, a).e_i : a \in A_i\}.$$

**Remark 2.1.** In [H0] the assumption that the sets  $A_i$  are disjoint is not restrictive: it is made only for simplifying the proof of Theorem 2.2 below. The assumption [H2] is not essential : it is made in order to avoid the use of relaxed controls.

Thanks to the Filippov implicit function lemma, see [19], we obtain:

**Theorem 2.1.** Let  $I$  be an interval of  $\mathbb{R}$  and  $\gamma : I \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  be a measurable function. Let  $K$  be a closed subset of  $\mathbb{R}^d \times A$  and  $\Psi : K \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  be continuous. Assume that  $\gamma(I) \subset \Psi(K)$ , then there is a measurable function  $\Phi : I \rightarrow K$  with

$$\Psi \circ \Phi(t) = \gamma(t) \quad \text{for a.a. } t \in I.$$

Let  $M$  denote the set:

$$M = \{(x, a); x \in \mathcal{S}, \quad a \in A_i \text{ if } x \in \mathcal{P}_i \setminus \Gamma, \text{ and } a \in \cup_{i=1}^N A_i \text{ if } x \in \Gamma\}. \quad (2.4)$$

The set  $M$  is closed. We also define the function  $f$  on  $M$  by

$$\forall (x, a) \in M, \quad f(x, a) = \begin{cases} f_i(x, a) & \text{if } x \in \mathcal{P}_i \setminus \Gamma, \\ f_i(x, a) & \text{if } x \in \Gamma \text{ and } a \in A_i. \end{cases} \quad (2.5)$$

**Remark 2.2.** The function  $f$  is well defined on  $M$ , because the sets  $A_i$  are disjoint, and is continuous on  $M$ .

Let  $\tilde{F}(x)$  be defined by

$$\tilde{F}(x) = \begin{cases} F_i(x) & \text{if } x \text{ belongs to the open half-plane } \mathcal{P}_i \setminus \Gamma \\ \cup_{i=1}^N F_i(x) & \text{if } x \in \Gamma. \end{cases}$$

For  $x \in \mathcal{S}$ , the set of admissible trajectories starting from  $x$  is

$$Y_x = \left\{ y_x \in \text{Lip}(\mathbb{R}^+; \mathcal{S}) : \begin{cases} \dot{y}_x(t) \in \tilde{F}(y_x(t)), & \text{for a.a. } t > 0, \\ y_x(0) = x, \end{cases} \right\}, \quad (2.6)$$

where  $\text{Lip}(\mathbb{R}^+; \mathcal{S})$  is the set of the Lipschitz continuous functions from  $\mathbb{R}_+$  to  $\mathcal{S}$ .

As in [6] and [2] the following statement holds :

**Theorem 2.2.** Assume [H0], [H1], [H2] and [H3]. Then

1. For any  $x \in \mathcal{S}$ ,  $Y_x$  is not empty.
2. For any  $x \in \mathcal{S}$ , for each trajectory  $y_x$  in  $Y_x$ , there exists a measurable function  $\Phi : [0, +\infty) \rightarrow M$ ,  $\Phi(t) = (\varphi_1(t), \varphi_2(t))$  with

$$(y_x(t), \dot{y}_x(t)) = (\varphi_1(t), f(\varphi_1(t), \varphi_2(t))), \quad \text{for a.a. } t,$$

which means in particular that  $y_x$  is a continuous representation of  $\varphi_1$ .

3. Almost everywhere on  $\{t : y_x(t) \in \Gamma\}$ ,  $f(y_x(t), \varphi_2(t)) \in \mathbb{R}e_0$ .

We introduce the set of admissible controlled trajectories starting from the initial datum  $x$  :

$$\mathcal{T}_x = \left\{ (y_x, \alpha) \in L_{\text{Loc}}^\infty(\mathbb{R}^+; M) : \begin{cases} y_x \in \text{Lip}(\mathbb{R}^+; \mathcal{S}), \\ y_x(t) = x + \int_0^t f(y_x(s), \alpha(s)) ds \quad \text{in } \mathbb{R}^+ \end{cases} \right\}. \quad (2.7)$$

**Remark 2.3.** *If two different half-planes are parallel to each other, say the half-planes  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , many other assumptions can be made on the dynamics and costs:*

- *a trivial case in which the assumptions [H1]-[H3] are satisfied is when the dynamics and costs are continuous at the origin, i.e.  $A_1 = A_2$ ;  $f_1$  and  $f_2$  are respectively the restrictions to  $\mathcal{P}_1 \times A_1$  and  $\mathcal{P}_2 \times A_2$  of a continuous and bounded function  $f_{1,2}$  defined in  $\mathbb{R}_0 \times \mathbb{R}_1 \times A_1$ , which is Lipschitz continuous with respect to the first variable;  $\ell_1$  and  $\ell_2$  are respectively the restrictions to  $\mathcal{P}_1 \times A_1$  and  $\mathcal{P}_2 \times A_2$  of a continuous and bounded function  $\ell_{1,2}$  defined in  $\mathbb{R}_0 \times \mathbb{R}_1 \times A_1$ .*
- *In this particular geometrical setting, one can allow some mixing (relaxation) at the vertex with several possible rules: More precisely, in [6, 7], Barles et al introduce several kinds of trajectories which stay at the interface: the regular trajectories are obtained by mixing outgoing dynamics from  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , whereas singular trajectories are obtained by mixing strictly ingoing dynamics from  $\mathcal{P}_1$  and  $\mathcal{P}_2$ . Two different value functions are obtained whether singular mixing is permitted or not.*

**The cost functional** The cost associated to the trajectory  $(y_x, \alpha) \in \mathcal{T}_x$  is

$$J(x; (y_x, \alpha)) = \int_0^\infty \ell(y_x(t), \alpha(t)) e^{-\lambda t} dt, \quad (2.8)$$

where  $\lambda > 0$  is a real number and the Lagrangian  $\ell$  is defined on  $M$  by

$$\forall (x, a) \in M, \quad \ell(x, a) = \begin{cases} \ell_i(x, a) & \text{if } x \in \mathcal{P}_i \setminus \Gamma, \\ \ell_i(x, a) & \text{if } x \in \Gamma \text{ and } a \in A_i. \end{cases} \quad (2.9)$$

**The value function** The value function of the infinite horizon optimal control problem is

$$v(x) = \inf_{(y_x, \alpha) \in \mathcal{T}_x} J(x; (y_x, \alpha)). \quad (2.10)$$

**Proposition 2.1.** *Assume [H0] and [H1]. We have the dynamic programming principle:*

$$\forall t \geq 0, \quad v(x) = \inf_{(y_x, \alpha) \in \mathcal{T}_x} \left\{ \int_0^t \ell(y_x(s), \alpha(s)) e^{-\lambda s} ds + e^{-\lambda t} v(y_x(t)) \right\}. \quad (2.11)$$

**Proposition 2.2.** *Assume [H0], [H1], [H2] and [H3]. Then the value function  $v$  is bounded and continuous on  $\mathcal{S}$ .*

Both propositions above are classical and can be proved with the same arguments as in [4].

## 2.2 The Hamilton-Jacobi equation

### 2.2.1 Test-functions

To define viscosity solutions on the irregular set  $\mathcal{S}$ , it is necessary to first define a class of admissible test-functions

**Definition 2.1.** *A function  $\varphi : \mathcal{S} \rightarrow \mathbb{R}$  is an admissible test-function if*

- *$\varphi$  is continuous in  $\mathcal{S}$*
- *for any  $j$ ,  $j = 1, \dots, N$ ,  $\varphi|_{\mathcal{P}_j} \in \mathcal{C}^1(\mathcal{P}_j)$ .*

The set of admissible test-functions is denoted by  $\mathcal{R}(\mathcal{S})$ . If  $\varphi \in \mathcal{R}(\mathcal{S})$ ,  $x \in \mathcal{S}$  and  $\zeta \in T_x(\mathcal{S})$ , let  $D\varphi(x, \zeta)$  be defined by

$$D\varphi(x, \zeta) = \begin{cases} D(\varphi|_{\mathcal{P}_i})(x) \cdot \zeta & \text{if } x \in \mathcal{P}_i \setminus \Gamma, \\ D(\varphi|_{\mathcal{P}_i})(x) \cdot \zeta & \text{if } x \in \Gamma \text{ and } \zeta \in \mathbb{R}e_0 \times \mathbb{R}e_i, \end{cases}$$

where for  $u, v \in \mathbb{R}e_0 \times \mathbb{R}e_i$ ,  $u \cdot v$  denotes the usual Euclidean scalar product in  $\mathbb{R}e_0 \times \mathbb{R}e_i$  and for  $x \in \Gamma$ ,  $D(\varphi|_{\mathcal{P}_i})(x) \cdot \xi$  is defined by

$$D(\varphi|_{\mathcal{P}_i})(x) \cdot \xi = \lim_{y \rightarrow x, y \in \mathcal{P}_i \setminus \Gamma} D(\varphi|_{\mathcal{P}_i})(y) \cdot \xi. \quad (2.12)$$

If  $\varphi \in \mathcal{R}(\mathcal{S})$ ,  $x \in \Gamma$  and  $\xi \in \mathbb{R}e_0$ , we will use also the notations  $D(\varphi|_{\Gamma})(x) \cdot \xi$  for the differential of  $\varphi|_{\Gamma}$  at the point  $x$  evaluated in  $\xi$ .

Other notations which will be useful are the following: for  $j \in \{1, \dots, N\}$ ,  $x \in \mathcal{P}_j$  and  $\phi \in \mathcal{R}(\mathcal{S})$

$$\partial_{x_j}(\phi|_{\mathcal{P}_j})(x) = \begin{cases} \lim_{h \rightarrow 0} \frac{\phi(x + he_j) - \phi(x)}{h} & \text{if } x \in \mathcal{P}_i \setminus \Gamma, \\ \lim_{h > 0} \frac{\phi(x + he_j) - \phi(x)}{h} & \text{if } x \in \Gamma, \end{cases} \quad (2.13)$$

and for  $x \in \Gamma$

$$\partial_{x_0}\phi(x) = \lim_{h \rightarrow 0} \frac{\phi(x + he_0) - \phi(x)}{h}. \quad (2.14)$$

**Remark 2.4.** If  $\varphi \in \mathcal{R}(\mathcal{S})$ ,  $x \in \Gamma$  and  $\xi \in \mathbb{R}$ , then for all  $i \in \{1, \dots, N\}$ ,

$$D(\varphi|_{\mathcal{P}_i})(x) \cdot \xi e_0 = D(\varphi|_{\Gamma})(x) \cdot \xi e_0 = \partial_{x_0}\phi(x) \cdot \xi.$$

Particularly, the tangential component of  $D(\varphi|_{\mathcal{P}_i})(x)$  is independent of  $i \in \{1, \dots, N\}$ .

**Property 2.1.** If  $\varphi = g \circ \psi$  with  $g \in \mathcal{C}^1(\mathbb{R})$  and  $\psi \in \mathcal{R}(\mathcal{S})$ , then  $\varphi \in \mathcal{R}(\mathcal{S})$  and for any  $x \in \mathcal{S}$ ,  $\zeta \in T_x(\mathcal{S})$

$$D\varphi(x, \zeta) = g'(\psi(x))D\psi(x, \zeta).$$

### 2.2.2 Vector fields

For  $i = 1, \dots, N$  and  $x$  in  $\Gamma$ , we denote by  $F_i^+(x)$  and  $\text{FL}_i^+(x)$  the sets

$$F_i^+(x) = F_i(x) \cap (\mathbb{R}e_0 \times \mathbb{R}^+e_i), \quad \text{FL}_i^+(x) = \text{FL}_i(x) \cap ((\mathbb{R}e_0 \times \mathbb{R}^+e_i) \times \mathbb{R}),$$

which are non empty thanks to assumption [H3]. Note that  $0 \in \cap_{i=1}^N F_i(x)$ . From assumption [H2], these sets are compact and convex. For  $x \in \mathcal{S}$ , the sets  $F(x)$  and  $\text{FL}(x)$  are defined by

$$F(x) = \begin{cases} F_i(x) & \text{if } x \text{ belongs to } \mathcal{P}_i \setminus \Gamma \\ \cup_{i=1, \dots, N} F_i^+(x) & \text{if } x \in \Gamma \end{cases}$$

and

$$\text{FL}(x) = \begin{cases} \text{FL}_i(x) & \text{if } x \text{ belongs to } \mathcal{P}_i \setminus \Gamma \\ \cup_{i=1, \dots, N} \text{FL}_i^+(x) & \text{if } x \in \Gamma. \end{cases}$$

### 2.2.3 Definition of viscosity solutions

We now introduce the definition of a viscosity solution of

$$\lambda u(x) + \sup_{(\zeta, \xi) \in \text{FL}(x)} \{-Du(x, \zeta) - \xi\} = 0 \quad \text{in } \mathcal{S}. \quad (2.15)$$

**Definition 2.2.** • An upper semi-continuous function  $u : \mathcal{S} \rightarrow \mathbb{R}$  is a subsolution of (2.15) in  $\mathcal{S}$  if for any  $x \in \mathcal{S}$ , any  $\varphi \in \mathcal{R}(\mathcal{S})$  s.t.  $u - \varphi$  has a local maximum point at  $x$ , then

$$\lambda u(x) + \sup_{(\zeta, \xi) \in \text{FL}(x)} \{-D\varphi(x, \zeta) - \xi\} \leq 0. \quad (2.16)$$

- A lower semi-continuous function  $u : \mathcal{S} \rightarrow \mathbb{R}$  is a supersolution of (2.15) if for any  $x \in \mathcal{S}$ , any  $\varphi \in \mathcal{R}(\mathcal{S})$  s.t.  $u - \varphi$  has a local minimum point at  $x$ , then

$$\lambda u(x) + \sup_{(\zeta, \xi) \in \text{FL}(x)} \{-D\varphi(x, \zeta) - \xi\} \geq 0. \quad (2.17)$$

- A continuous function  $u : \mathcal{S} \rightarrow \mathbb{R}$  is a viscosity solution of (2.15) in  $\mathcal{S}$  if it is both a viscosity subsolution and a viscosity supersolution of (2.15) in  $\mathcal{S}$ .

**Remark 2.5.** At  $x \in \mathcal{P}_i \setminus \Gamma$ , the notion of sub, respectively super-solution in Definition 2.2 is equivalent to the standard definition of viscosity sub, respectively super-solution of

$$\lambda u(x) + \sup_{a \in A_i} \{-f_i(x, a) \cdot Du(x) - \ell_i(x, a)\} = 0.$$

## 2.2.4 Hamiltonians

We define the Hamiltonian  $H_i : \mathcal{P}_i \times (\mathbb{R}e_0 \times \mathbb{R}e_i) \rightarrow \mathbb{R}$  by

$$H_i(x, p) = \max_{a \in A_i} (-p \cdot f_i(x, a) - \ell_i(x, a)), \quad (2.18)$$

and the Hamiltonian  $H_\Gamma : \Gamma \times \left(\prod_{i=1, \dots, N} (\mathbb{R}e_0 \times \mathbb{R}e_i)\right) \rightarrow \mathbb{R}$  by

$$H_\Gamma(x, p_1, \dots, p_N) = \max_{i=1, \dots, N} H_i^+(x, p_i), \quad (2.19)$$

where the Hamiltonian  $H_i^+ : \mathcal{P}_i \times (\mathbb{R}e_0 \times \mathbb{R}e_i) \rightarrow \mathbb{R}$  is defined by

$$H_i^+(x, p) = \max_{a \in A_i \text{ s.t. } f_i(x, a) \cdot e_i \geq 0} (-p \cdot f_i(x, a) - \ell_i(x, a)). \quad (2.20)$$

We also define what may be called the tangential Hamiltonian at  $\Gamma$ ,  $H_\Gamma^T : \Gamma \times \mathbb{R}e_0 \rightarrow \mathbb{R}$ , by

$$H_\Gamma^T(x, p) = \max_{i=1, \dots, N} H_{\Gamma, i}^T(x, p), \quad (2.21)$$

where the Hamiltonian  $H_{\Gamma, i}^T : \Gamma \times \mathbb{R}e_0 \rightarrow \mathbb{R}$  is defined by

$$H_{\Gamma, i}^T(x, p) = \max_{a \in A_i \text{ s.t. } f_i(x, a) \cdot e_i = 0} (-f_i(x, a) \cdot p - \ell_i(x, a)). \quad (2.22)$$

The following definitions are equivalent to Definition 2.2:

**Definition 2.3.** • An upper semi-continuous function  $u : \mathcal{S} \rightarrow \mathbb{R}$  is a subsolution of (2.15) in  $\mathcal{S}$  if for any  $x \in \mathcal{S}$ , any  $\varphi \in \mathcal{R}(\mathcal{S})$  s.t.  $u - \varphi$  has a local maximum point at  $x$ , then

$$\begin{aligned} \lambda u(x) + H_i(x, D(\varphi|_{\mathcal{P}_i})(x)) &\leq 0 & \text{if } x \in \mathcal{P}_i \setminus \Gamma, \\ \lambda u(x) + H_\Gamma(x, D(\varphi|_{\mathcal{P}_1})(x), \dots, D(\varphi|_{\mathcal{P}_N})(x)) &\leq 0 & \text{if } x \in \Gamma. \end{aligned} \quad (2.23)$$

- A lower semi-continuous function  $u : \mathcal{S} \rightarrow \mathbb{R}$  is a supersolution of (2.15) if for any  $x \in \mathcal{S}$ , any  $\varphi \in \mathcal{R}(\mathcal{S})$  s.t.  $u - \varphi$  has a local minimum point at  $x$ , then

$$\begin{aligned} \lambda u(x) + H_i(x, D(\varphi|_{\mathcal{P}_i})(x)) &\geq 0 & \text{if } x \in \mathcal{P}_i \setminus \Gamma, \\ \lambda u(x) + H_\Gamma(x, D(\varphi|_{\mathcal{P}_1})(x), \dots, D(\varphi|_{\mathcal{P}_N})(x)) &\geq 0 & \text{if } x \in \Gamma. \end{aligned} \quad (2.24)$$

The Hamiltonian  $H_i$  are continuous with respect to  $x \in \mathcal{P}_i$ , convex with respect to  $p$ . Moreover, if  $x$  belongs to  $\Gamma$ , the function  $p \mapsto H_i(x, p)$  is coercive, i.e.  $\lim_{|p| \rightarrow +\infty} H_i(x, p) = +\infty$  from the controllability assumption [H3].

The following lemma is the counterpart of Lemma 2.1 in [2].



**Lemma 2.1.** Assume [H0], [H1], [H2] and [H3]. Take  $i \in \{1, \dots, N\}$ ,  $x \in \Gamma$  and  $p \in \mathbb{R}e_0 \times \mathbb{R}e_i$ . Let  $\varphi_{i,x,p} : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by  $\varphi_{i,x,p}(\delta) = H_i(x, p + \delta e_i)$ . We denote by  $\Delta_{i,x,p}$  the set

$$\Delta_{i,x,p} = \{\delta \in \mathbb{R} \text{ s.t. } \varphi_{i,x,p}(\delta) = \min_{d \in \mathbb{R}}(\varphi_{i,x,p}(d))\}. \quad (2.25)$$

1. The set  $\Delta_{i,x,p}$  is not empty.
2.  $\delta \in \Delta_{i,x,p}$  if and only if there exists  $a^* \in A_i$  such that  $f_i(x, a^*).e_i = 0$  and  $\varphi_{i,x,p}(\delta) = -f_i(x, a^*).p - \ell_i(x, a^*)$ .
3. For any  $x \in \Gamma$ ,  $p = p_0 e_0 + p_i e_i$ , with  $p_0, p_i \in \mathbb{R}$  and  $\delta \in \Delta_{i,x,p}$  we have

$$H_i(x, p + \delta e_i) = H_i^+(x, p + \delta e_i) = H_{\Gamma,i}^T(x, p_0 e_0). \quad (2.26)$$

4. For any  $x \in \Gamma$ ,  $p = p_0 e_0 + p_i e_i$ , with  $p_0, p_i \in \mathbb{R}$  and  $\delta \geq \min\{d : d \in \Delta_{i,x,p}\}$  we have

$$H_i^+(x, p + \delta e_i) = H_{\Gamma,i}^T(x, p_0 e_0). \quad (2.27)$$

*Proof.* Point 1 is easy, because the Hamiltonian  $H_i$  is continuous and coercive with respect to  $p$ . The function  $\varphi_{i,x,p}$  reaches its minimum at  $\delta$  if and only if  $0 \in \partial\varphi_{i,x,p}(\delta)$ . The subdifferential of  $\varphi_{i,x,p}$  at  $\delta$  is characterized by

$$\partial\varphi_{i,x,p}(\delta) = \overline{\text{co}}\{-f_i(x, a).e_i; a \in A_i \text{ s.t. } \varphi_{i,x,p}(\delta) = -f_i(x, a).(p + \delta e_i) - \ell_i(x, a)\},$$

see [25]. But from [H2],

$$\{(f_i(x, a), \ell_i(x, a)); a \in A_i \text{ s.t. } \varphi_{i,x,p}(\delta) = -f_i(x, a).(p + \delta e_i) - \ell_i(x, a)\}$$

is compact and convex. Hence,

$$\partial\varphi_{i,x,p}(\delta) = \{-f_i(x, a).e_i; a \in A_i \text{ s.t. } \varphi_{i,x,p}(\delta) = -f_i(x, a).(p + \delta e_i) - \ell_i(x, a)\}.$$

Therefore,  $0 \in \partial\varphi_{i,x,p}(\delta)$  if and only if there exists  $a^* \in A_i$  such that  $f_i(x, a^*).e_i = 0$  and  $\varphi_{i,x,p}(\delta) = -f_i(x, a^*).p - \ell_i(x, a^*)$ . We have proved point 2.

Points 3 is a direct consequence of point 2. Point 4 is a consequence of point 3 and of the decreasing character of the function  $d \mapsto H_i^+(x, p + d e_i)$ .  $\square$

**Remark 2.6.** The conclusions of Lemma 2.1 hold if we replace [H3] with  $[\tilde{\text{H3}}]$ . Indeed, we actually just need that the Hamiltonian  $H_i$  be coercive with respect to  $p_i$ , where  $p_i = p.e_i$ .

### 2.2.5 Existence

**Theorem 2.3.** Assume [H0], [H1], [H2] and [H3]. The value function  $v$  defined in (2.10) is a bounded viscosity solution of (2.15) in  $\mathcal{S}$ .

The proof of Theorem 2.3 is made in several steps: the first step consists of proving that the value function is a viscosity solution of a Hamilton-Jacobi equation with a more general definition of the Hamiltonian: for that, we introduce larger relaxed vector fields: for  $x \in \mathcal{S}$ ,

$$\tilde{f}(x) = \left\{ \eta \in \mathbb{R}^d : \begin{array}{l} \exists (y_{x,n}, \alpha_n)_{n \in \mathbb{N}}, \\ (y_{x,n}, \alpha_n) \in \mathcal{T}_x, \text{ s.t.} \\ \exists (t_n)_{n \in \mathbb{N}} \end{array} \left| \begin{array}{l} t_n \rightarrow 0^+ \text{ and} \\ \lim_{n \rightarrow \infty} \frac{1}{t_n} \int_0^{t_n} f(y_{x,n}(t), \alpha_n(t)) dt = \eta \end{array} \right. \right\}$$

and

$$\tilde{f}\ell(x) = \left\{ (\eta, \mu) \in \mathbb{R}^d \times \mathbb{R} : \begin{array}{l} \exists (y_{x,n}, \alpha_n)_{n \in \mathbb{N}}, \\ (y_{x,n}, \alpha_n) \in \mathcal{T}_x, \text{ s.t.} \\ \exists (t_n)_{n \in \mathbb{N}} \end{array} \left| \begin{array}{l} t_n \rightarrow 0^+, \\ \lim_{n \rightarrow \infty} \frac{1}{t_n} \int_0^{t_n} f(y_{x,n}(t), \alpha_n(t)) dt = \eta, \\ \lim_{n \rightarrow \infty} \frac{1}{t_n} \int_0^{t_n} \ell(y_{x,n}(t), \alpha_n(t)) dt = \mu \end{array} \right. \right\};$$

where  $\mathcal{T}_x$  is the set of admissible controlled trajectories starting from the point  $x$  which was introduced in (2.7).

**Proposition 2.3.** Assume [H0], [H1], [H2] and [H3]. The value function  $v$  defined in (2.10) is a viscosity solution of

$$\lambda u(x) + \sup_{(\zeta, \xi) \in \tilde{f\ell}(x)} \{-Du(x, \zeta) - \xi\} = 0 \quad \text{in } \mathcal{S}, \quad (2.28)$$

where the definition of viscosity solution is exactly the same as in Definition 2.2, replacing  $\text{FL}(x)$  with  $\tilde{f\ell}(x)$ .

*Proof.* See [1].  $\square$

The second step consists of proving that for any  $\varphi \in \mathcal{R}(\mathcal{S})$  and  $x \in \mathcal{S}$ ,  $\sup_{(\zeta, \xi) \in \text{FL}(x)} \{-D\varphi(x, \zeta) - \xi\}$  and  $\sup_{(\zeta, \xi) \in \tilde{f\ell}(x)} \{-D\varphi(x, \zeta) - \xi\}$  are equal. This is a consequence of the following lemma.

**Lemma 2.2.** For any  $x \in \mathcal{S}$ ,

$$\begin{aligned} \tilde{f\ell}(x) &= \text{FL}(x) && \text{if } x \in \mathcal{S} \setminus \Gamma, \\ \tilde{f\ell}(x) &= \bigcup_{i=1, \dots, N} \overline{\text{co}} \left\{ \text{FL}_i^+(x) \cup \bigcup_{j \neq i} \left( \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R}) \right) \right\} && \text{if } x \in \Gamma. \end{aligned}$$

*Proof.* The proof being a bit long, we postpone it to the appendix A.  $\square$

**Lemma 2.3.** Assume [H0], [H1], [H2] and [H3]. For any function  $\varphi \in \mathcal{R}(\mathcal{S})$  and  $x \in \mathcal{S}$ ,

$$\sup_{(\zeta, \xi) \in \tilde{f\ell}(x)} \{-D\varphi(x, \zeta) - \xi\} = \max_{(\zeta, \xi) \in \text{FL}(x)} \{-D\varphi(x, \zeta) - \xi\}. \quad (2.29)$$

*Proof.* For  $x \in \mathcal{S} \setminus \Gamma$  there is nothing to prove because  $\text{FL}(x) = \tilde{f\ell}(x)$ . If  $x \in \Gamma$  we can prove that  $\text{FL}(x) \subset \tilde{f\ell}(x)$  for any  $x \in \Gamma$ , in the same way as in [1]. Hence

$$\max_{(\zeta, \xi) \in \text{FL}(x)} \{-D\varphi(x, \zeta) - \xi\} \leq \sup_{(\zeta, \xi) \in \tilde{f\ell}(x)} \{-D\varphi(x, \zeta) - \xi\}.$$

From the piecewise linearity of the function  $(\zeta, \mu) \mapsto -D\varphi(x, \zeta) - \mu$ , we infer that

$$\begin{aligned} & \sup_{(\zeta, \mu) \in \overline{\text{co}} \left\{ \text{FL}_i^+(x) \cup \bigcup_{j \neq i} \left( \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R}) \right) \right\}} (-D\varphi(x, \zeta) - \mu) \\ &= \max \left( \max_{(\zeta, \mu) \in \text{FL}_i^+(x)} (-D\varphi(x, \zeta) - \mu), \max_{j \neq i} \left( \max_{(\zeta, \mu) \in \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R})} (-D\varphi(x, \zeta) - \mu) \right) \right) \\ &\leq \max_{j=1, \dots, N} \max_{(\zeta, \mu) \in \text{FL}_j^+(x)} (-D\varphi(x, \zeta) - \mu) = \max_{(\zeta, \xi) \in \text{FL}(x)} \{-D\varphi(x, \zeta) - \xi\}. \end{aligned}$$

We conclude by using Lemma 2.2.  $\square$

### 2.3 Properties of viscosity sub and supersolutions

In this part, we study sub and supersolutions of (2.15), transposing ideas coming from Barles-Briani-Chasseigne [6, 7] to the present context.

**Property 2.2.** Assume [H0] and [H3]. Then, there exists  $R > 0$  a positive real number such that for all  $i = 1, \dots, N$  and  $x \in B(\Gamma, R) \cap \mathcal{P}_i$

$$B(x, \frac{\delta}{2}) \cap (\mathbb{R}e_0 \times \mathbb{R}e_i) \subset F_i(x). \quad (2.30)$$

**Remark 2.7.** This property means that the controllability assumption [H3], which focuses on  $\Gamma$ , holds in a neighborhood of  $\Gamma$  thanks to the continuity properties of the functions  $f_i$ ,  $i \in \{1, \dots, N\}$ .

**Lemma 2.4.** Assume [H0], [H1], [H2] and [H3]. Let  $R > 0$  be as in (2.30). For any bounded viscosity subsolution  $u$  of (2.15), there exists a constant  $C^* > 0$  such that  $u$  is a viscosity subsolution of

$$|Du(x)| \leq C^* \quad \text{in } B(\Gamma, R) \cap \mathcal{S}, \quad (2.31)$$

i.e. for any  $x \in B(\Gamma, R) \cap \mathcal{S}$  and  $\phi \in \mathcal{R}(\mathcal{S})$  such that  $u - \phi$  has a local maximum point at  $x$ ,

$$|D(\phi|_{\mathcal{P}_i})(x)| \leq C^* \quad \text{if } x \in (B(\Gamma, R) \cap \mathcal{P}_i) \setminus \Gamma, \quad (2.32)$$

$$\max_{i=1, \dots, N} \{|\partial_{x_0} \phi(x)| + [\partial_{x_i}(\phi|_{\mathcal{P}_i})(x)]_-\} \leq C^* \quad \text{if } x \in \Gamma, \quad (2.33)$$

where  $\partial_{x_j}(\phi|_{\mathcal{P}_j})(x)$  and  $\partial_{x_0} \phi(x)$  are defined in (2.13) and (2.14) and  $[\cdot]_-$  denote the negative part function, i.e. for  $x \in \mathbb{R}$ ,  $[x]_- = \max\{0, -x\}$ .

*Proof.* Let  $M_u$  (resp  $M_\ell$ ) be an upper bound on  $|u|$  (resp.  $\ell_j$  for all  $j = 1, \dots, N$ ). The viscosity inequality (2.23) yields that

$$H_i(x, D(\phi|_{\mathcal{P}_i})(x)) \leq \lambda M_u \quad \text{if } x \in (B(\Gamma, R) \cap \mathcal{P}_i) \setminus \Gamma, \quad (2.34)$$

$$H_\Gamma(x, D(\phi|_{\mathcal{P}_1})(x), \dots, D(\phi|_{\mathcal{P}_N})(x)) \leq \lambda M_u \quad \text{if } x \in \Gamma. \quad (2.35)$$

From the controllability in  $B(\Gamma, R) \cap \mathcal{P}_i$ , we see that  $H_i$  is coercive with respect to its second argument uniformly in  $x \in B(\Gamma, R) \cap \mathcal{P}_i$ . More precisely we have that  $H_i(x, p) \geq \frac{\delta}{2}|p| - M_\ell$ . Thus, if  $x \in (B(\Gamma, R) \cap \mathcal{P}_i) \setminus \Gamma$ , from (2.34), we have  $|D(\phi|_{\mathcal{P}_i})(x)| \leq C^*$  with  $C^* = 2\frac{\lambda M_u + M_\ell}{\delta}$ . If  $x \in \Gamma$ , considering the controls  $a_0^+, a_0^-, a_i^+$  and  $a_i^0 \in A_i$  such that  $f_i(x, a_0^\pm) = \pm \delta e_0$ ,  $f_i(x, a_i^+) = \delta e_i$  and  $f_i(x, a_i^0) = 0$  we can prove that  $H_i^+(x, p_0 e_0 + p_i e_i) \geq \frac{\delta}{2}(|p_0| + |p_i|_-) - M_\ell$ . Then, the viscosity inequality (2.35) yields (2.33) with the same constant  $C^* = 2\frac{\lambda M_u + M_\ell}{\delta}$ .  $\square$

The following lemma gives us an explicit expression for the geodesic distance which will be convenient in future calculations.

**Lemma 2.5.** Let  $x = x_0 e_0 + x_i e_i \in \mathcal{P}_i$  and  $y = y_0 e_0 + y_j e_j \in \mathcal{P}_j$ . Then,

$$d(x, y) = \begin{cases} [(x_0 - y_0)^2 + (x_i - y_i)^2]^{\frac{1}{2}} & \text{if } i = j, \\ [(x_0 - y_0)^2 + (x_i + y_j)^2]^{\frac{1}{2}} & \text{if } i \neq j. \end{cases} \quad (2.36)$$

**Lemma 2.6.** Assume [H0], [H1], [H2] and [H3]. Any bounded viscosity subsolution  $u$  of (2.15) is Lipschitz continuous in a neighborhood of  $\Gamma$ , i.e there exists some strictly positive number  $r$  such that  $u$  is Lipschitz continuous on  $B(\Gamma, r) \cap \mathcal{S}$ , where  $B(\Gamma, r)$  denotes the set  $\{y \in \mathbb{R}^d : \text{dist}(y, \Gamma) < r\}$ .

*Proof.* We adapt the proof of H.Ishii, see [17].

Take  $R$  as in (2.30), fix  $z \in B(\Gamma, R) \cap \mathcal{S}$  and set  $r = (R - \text{dist}(z, \Gamma))/4$ . Fix any  $y \in \mathcal{S}$  such that  $d(y, z) < r$ . It can be checked that for any  $x \in \mathcal{S}$ , if  $d(x, y) < 3r$  then  $\text{dist}(x, \Gamma) < R$ . Choose a function  $f \in C^1([0, 3r])$  such that  $f(t) = t$  in  $[0, 2r]$ ,  $f'(t) \geq 1$  for all  $t \in [0, 3r]$  and  $\lim_{t \rightarrow 3r} f(t) = +\infty$ . Fix any  $\varepsilon > 0$ . We are going to show that

$$u(x) \leq u(y) + (C^* + \varepsilon)f(d(x, y)), \quad \forall x \in \mathcal{S} \text{ such that } d(x, y) < 3r, \quad (2.37)$$

where  $C^*$  is the constant in Lemma 2.4. Let us proceed by contradiction. Assume that (2.37) is not true. According to the properties of  $f$ , the function  $x \mapsto u(x) - u(y) - (C^* + \varepsilon)f(d(x, y))$  admits a maximum  $\xi \in B(y, 3r) \cap \mathcal{S}$ . However, since we assumed that (2.37) is not true, necessarily  $\xi \neq y$ . Consequently, the function  $\psi : \mathcal{S} \rightarrow \mathbb{R}$ ,  $x \mapsto (C^* + \varepsilon)f(d(x, y))$  is an appropriate test function in a neighborhood of  $\xi$  which can be used as test function in the viscosity inequality (2.31), satisfied by  $u$  from Lemma 2.4. For the calculations, we need to distinguish several cases. Assume that  $y = y_0 e_0 + y_i e_i \in \mathcal{P}_i$ .

1. **If**  $\xi = \xi_0 e_0 + \xi_i e_i \in \mathcal{P}_i \setminus \Gamma$  : i.e.  $\xi$  and  $y$  belong to the same half-plane  $\mathcal{P}_i$ . Then, from (2.36) in Lemma 2.5, we have  $D(\psi|_{\mathcal{P}_i})(\xi) = (C^* + \varepsilon)f'(d(\xi, y))\frac{\xi - y}{d(\xi, y)}$  and (2.32) in Lemma 2.4 gives us

$$(C^* + \varepsilon)f'(d(\xi, y))\frac{|\xi - y|}{d(\xi, y)} \leq C^*. \quad (2.38)$$

Since  $\frac{|\xi - y|}{d(\xi, y)} = 1$  and  $f'(t) \geq 1$  for all  $t \in [0, 3r)$ , (2.38) leads to a contradiction.

2. **If**  $\xi = \xi_0 e_0 + \xi_j e_j \in \mathcal{P}_j \setminus \Gamma$  **with**  $j \neq i$  : i.e.  $\xi$  and  $y$  belong to different half-planes, respectively  $\mathcal{P}_j$  and  $\mathcal{P}_i$ . Then, from (2.36) in Lemma 2.5, we have  $D(\psi|_{\mathcal{P}_j})(\xi) = (C^* + \varepsilon)f'(d(\xi, y))\frac{(\xi_0 - y_0)e_0 + (\xi_j + y_i)e_j}{d(\xi, y)}$  and (2.32) in Lemma 2.4 gives us

$$(C^* + \varepsilon)f'(d(\xi, y))\frac{|(\xi_0 - y_0)e_0 + (\xi_j + y_i)e_j|}{d(\xi, y)} \leq C^*. \quad (2.39)$$

Since  $\frac{|(\xi_0 - y_0)e_0 + (\xi_j + y_i)e_j|}{d(\xi, y)} = 1$  and  $f'(t) \geq 1$  for all  $t \in [0, 3r)$ , (2.39) leads to a contradiction.

3. **If**  $\xi = \xi_0 e_0 \in \Gamma$  : In this case,  $\xi$  and  $y$  belong to the same half-plane, but we have to deal with (2.33) in Lemma 2.4. For  $i \in \{1, \dots, N\}$  be such that  $y \in \mathcal{P}_i$ , from (2.36) in Lemma 2.5 we have

$$\partial_{x_i}(\psi|_{\mathcal{P}_i})(\xi) = (C^* + \varepsilon)f'(d(\xi, y))\frac{-y_i}{d(\xi, y)} \leq 0,$$

and

$$\partial_{x_0}\psi(\xi) = (C^* + \varepsilon)g'(d(\xi, y))\frac{\xi_0 - y_0}{d(\xi, y)}.$$

Then, (2.33) in Lemma 2.4 gives us

$$(C^* + \varepsilon)f'(d(\xi, y))\frac{|\xi_0 - y_0| + y_i}{d(\xi, y)} \leq C^*. \quad (2.40)$$

Since  $\frac{|\xi_0 - y_0| + y_i}{d(\xi, y)} \geq 1$  and  $f'(t) \geq 1$  for all  $t \in [0, 3r)$ , (2.40) leads to a contradiction.

This conclude the proof of (2.37). Remark that if  $d(x, z) < r$  then  $d(x, y) < 2r$  and  $f(d(x, y)) = d(x, y)$ . Then, (2.37) yields that

$$u(x) \leq u(y) + (C^* + \varepsilon)d(x, y), \quad \forall x, y \in \mathcal{S} \text{ s.t } d(x, z) < r, d(y, z) < r.$$

By symmetry, we get

$$|u(x) - u(y)| \leq (C^* + \varepsilon)d(x, y), \quad \forall x, y \in \mathcal{S} \text{ s.t } d(x, z) < r, d(y, z) < r,$$

and by letting  $\varepsilon$  tend to zero, we get

$$|u(x) - u(y)| \leq C^*d(x, y), \quad \forall x, y \in \mathcal{S} \text{ s.t } d(x, z) < r, d(y, z) < r. \quad (2.41)$$

Now, for two arbitrary points  $x, y$  in  $\mathcal{S} \cap B(\Gamma, R)$ , we take  $r = \frac{1}{4} \min(R - \text{dist}(x, \Gamma), R - \text{dist}(y, \Gamma))$  and choose a finite sequence  $(z_j)_{j=1, \dots, M} \in \mathcal{G}$  belonging to the geodesic between  $x$  and  $y$ , such that  $z_1 = x$ ,  $z_M = y$ ,  $d(z_i, z_{i+1}) < r$  for all  $i = 1, \dots, M - 1$  and  $\sum_{i=1}^{M-1} d(z_i, z_{i+1}) = d(x, y)$ . From (2.41), we get that

$$|u(x) - u(y)| \leq C^*d(x, y), \quad \forall x, y \in \mathcal{S} \cap B(\Gamma, R).$$

□

An important consequence of this lemma is the following result.

**Lemma 2.7.** Assume [H0], [H1], [H2] and [H3]. Let  $u$  be a bounded viscosity subsolution of (2.15) and  $\varphi : \Gamma \rightarrow \mathbb{R}$  a  $C^1$ -function. Then, for any local maximum point  $\bar{x}$  in  $\Gamma$  of  $u - \varphi$  on  $\Gamma$ , one has

$$\lambda u(\bar{x}) + H_\Gamma^T(\bar{x}, D(\varphi|_\Gamma)(\bar{x})) \leq 0.$$

*Proof.* Let  $\varphi : \Gamma \rightarrow \mathbb{R}$  be a  $C^1$ -function and  $\bar{x} \in \Gamma$  be a local maximum point of  $(u - \varphi)|_\Gamma$  on  $\Gamma$ . Since  $u$  is a subsolution of (2.15), according to Lemma 2.6, the function  $u$  is Lipschitz continuous on  $B(\Gamma, r)$  for some positive real  $r$ , with a Lipschitz constant  $L_{u,r}$ . We introduce the function  $\bar{\varphi}$  defined on  $\mathcal{S}$  by  $\bar{\varphi}(x_0 e_0 + x_i e_i) = \varphi(x_0 e_0) + L_{u,r} x_i$ . By construction, the function  $\bar{\varphi}$  belongs to  $\mathcal{R}(\mathcal{S})$  and  $u - \bar{\varphi}$  admits a local maximum at the point  $\bar{x}$  on  $\mathcal{S}$ . Then, since  $u$  is a viscosity subsolution of (2.15), we see that

$$\lambda u(\bar{x}) + H_\Gamma(\bar{x}, D(\bar{\varphi}|_{\mathcal{P}_1})(\bar{x}), \dots, D(\bar{\varphi}|_{\mathcal{P}_N})(\bar{x})) \leq 0,$$

which implies that  $\lambda u(\bar{x}) + H_\Gamma^T(\bar{x}, D(\varphi|_\Gamma)(\bar{x})) \leq 0$ .  $\square$

**Remark 2.8.** The conclusion of Lemma 2.7 holds if we replace [H3] with the assumption that the subsolution  $u$  is Lipschitz continuous.

The following lemma can be found in [6, 7] in a different context:

**Lemma 2.8.** Assume [H0], [H1], [H2] and [H3]. Let  $v : \mathcal{S} \rightarrow \mathbb{R}$  be a viscosity supersolution of (2.15) in  $\mathcal{S}$  and  $w : \mathcal{S} \rightarrow \mathbb{R}$  be a continuous viscosity subsolution of (2.15) in  $\mathcal{S}$ . Then if  $x \in \mathcal{P}_i \setminus \Gamma$ , we have for all  $t > 0$ ,

$$v(x) \geq \inf_{\alpha_i(\cdot), \theta_i} \left( \int_0^{t \wedge \theta_i} \ell_i(y_x^i(s), \alpha_i(s)) e^{-\lambda s} ds + v(y_x^i(t \wedge \theta_i)) e^{-\lambda(t \wedge \theta_i)} \right), \quad (2.42)$$

and

$$w(x) \leq \inf_{\alpha_i(\cdot), \theta_i} \left( \int_0^{t \wedge \theta_i} \ell_i(y_x^i(s), \alpha_i(s)) e^{-\lambda s} ds + w(y_x^i(t \wedge \theta_i)) e^{-\lambda(t \wedge \theta_i)} \right), \quad (2.43)$$

where  $\alpha_i \in L^\infty(0, \infty; A_i)$ ,  $y_x^i$  is the solution of  $y_x^i(t) = x + \left( \int_0^t f_i(y_x^i(s), \alpha_i(s)) ds \right)$  and  $\theta_i$  is such that  $y_x^i(\theta_i) \in \Gamma$  and  $\theta_i$  lies in  $[\tau_i, \bar{\tau}_i]$ , where  $\tau_i$  is the exit time of  $y_x^i$  from  $\mathcal{P}_i \setminus \Gamma$  and  $\bar{\tau}_i$  is the exit time of  $y_x^i$  from  $\mathcal{P}_i$ .

*Proof.* See [6].  $\square$

**Remark 2.9.** The conclusions of Lemma 2.8 hold if we replace [H3] with  $[\tilde{\text{H}}3]$ . See [7].

The following theorem is reminiscent of Theorem 3.3 in [6]:

**Theorem 2.4.** Assume [H0], [H1], [H2] and [H3]. Let  $v : \mathcal{S} \rightarrow \mathbb{R}$  be a viscosity supersolution of (2.15), bounded from below by  $-c|x| - C$  for two positive numbers  $c$  and  $C$ . Let  $\phi \in \mathcal{R}(\mathcal{S})$  and  $\bar{x} \in \Gamma$  be such that  $v - \phi$  has a local minimum point at  $\bar{x}$ . Then, either [A] or [B] below is true:

[A] There exists  $\eta > 0$ ,  $i \in \{1, \dots, N\}$  and a sequence  $x_k \in \mathcal{P}_i \setminus \Gamma$ ,  $\lim_{k \rightarrow +\infty} x_k = \bar{x}$  such that  $\lim_{k \rightarrow +\infty} v(x_k) = v(\bar{x})$  and for each  $k$ , there exists a control law  $\alpha_i^k$  such that the corresponding trajectory  $y_{x_k}(s) \in \mathcal{P}_i$  for all  $s \in [0, \eta]$  and

$$v(x_k) \geq \int_0^\eta \ell_i(y_{x_k}(s), \alpha_i^k(s)) e^{-\lambda s} ds + v(y_{x_k}(\eta)) e^{-\lambda \eta}. \quad (2.44)$$

[B]

$$\lambda v(\bar{x}) + H_\Gamma^T(\bar{x}, D(\phi|_\Gamma)(\bar{x})) \geq 0. \quad (2.45)$$

*Proof.* For any  $i$  in  $\{1, \dots, N\}$  we consider the function  $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$  defined as follows

$$\varphi_i(d) = H_i(\bar{x}, D(\phi|_{\mathcal{P}_i})(\bar{x}) + de_i)$$

For  $i$  in  $\{1, \dots, N\}$ , let  $q_i$  be a real number such that  $\varphi_i(q_i) = \min_{d \in \mathbb{R}} \{\varphi_i(d)\}$ . We can already remark that according to Lemma 2.1 we have,

$$H_\Gamma(\bar{x}, D(\phi|_{\mathcal{P}_1})(\bar{x}) + q_1 e_1, \dots, D(\phi|_{\mathcal{P}_N})(\bar{x}) + q_N e_N) = H_\Gamma^T(\bar{x}, D(\phi|_\Gamma)(\bar{x})). \quad (2.46)$$

Consider the function

$$\psi_\varepsilon(x) = v(x) - \phi(x) - q_i x_i + \frac{x_i^2}{\varepsilon^2} \quad \text{if } x \in \mathcal{P}_i,$$

defined on  $\mathcal{S}$ , where  $x_i = x \cdot e_i$ . Changing  $\phi(x)$  into  $\phi(x) - |x - \bar{x}|^2$  if necessary, we can assume that  $\bar{x}$  is a strict local minimum point of  $v - \phi$ . Then, standard arguments show that for  $\varepsilon$  small enough, the function  $\psi_\varepsilon$  reaches its minimum close to  $\bar{x}$ , and that any sequence of such minimum points  $x_\varepsilon$  converges to  $\bar{x}$  and satisfies  $v(x_\varepsilon)$  converges to  $v(\bar{x})$ .

Up to the extraction of a subsequence, we can make out two cases.

1. **If  $x_\varepsilon \in \Gamma$  :** Then, since  $v$  is a viscosity supersolution of (2.15), we have

$$\lambda v(x_\varepsilon) + H_\Gamma(x_\varepsilon, D(\phi|_{\mathcal{P}_1})(x_\varepsilon) + q_1 e_1, \dots, D(\phi|_{\mathcal{P}_N})(x_\varepsilon) + q_N e_N) \geq 0,$$

and letting  $\varepsilon$  tend to 0, we obtain

$$\lambda v(\bar{x}) + H_\Gamma(\bar{x}, D(\phi|_{\mathcal{P}_1})(\bar{x}) + q_1 e_1, \dots, D(\phi|_{\mathcal{P}_N})(\bar{x}) + q_N e_N) \geq 0. \quad (2.47)$$

Then, using (2.46) we deduce from (2.47) that

$$\lambda v(\bar{x}) + H_\Gamma^T(\bar{x}, D(\phi|_\Gamma)(\bar{x})) \geq 0,$$

hence  $[B]$ .

2. **If  $x_\varepsilon \in \mathcal{P}_i \setminus \Gamma$  for some  $i \in \{1, \dots, N\}$  :** We skip the writing of this case which is treated as in the corresponding case in the proof of [2, Theorem 4.1], using the superoptimality (2.42) of Lemma 2.8.

□

**Remark 2.10.** The conclusions of Theorem 2.4 hold if we replace [H3] with  $[\tilde{H}3]$ . Indeed, the proof is based on Lemma 2.1 and Lemma 2.8 which stay true if we replace [H3] with  $[\tilde{H}3]$ .

## 2.4 Comparison principle and uniqueness

**Lemma 2.9.** Assume [H0], [H1], [H2] and [H3]. Let  $u : \mathcal{S} \rightarrow \mathbb{R}$  be a bounded continuous viscosity subsolution of (2.15). Let  $(\rho_\varepsilon)_\varepsilon$  be a sequence of mollifiers defined on  $\mathbb{R}$  as follows

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon} \rho\left(\frac{x}{\varepsilon}\right),$$

where

$$\rho \in C^\infty(\mathbb{R}, \mathbb{R}^+), \quad \int_{\mathbb{R}} \rho(x) dx = 1 \quad \text{and} \quad \text{supp}(\rho) = [-1, 1].$$

We consider the function  $u_\varepsilon$  defined on  $\mathcal{S}$  by

$$u_\varepsilon(x) = u * \rho_\varepsilon(x) = \int_{\mathbb{R}} u((x_0 - \tau)e_0 + x') \rho_\varepsilon(\tau) d\tau, \quad \text{if } x = x_0 e_0 + x',$$

where the decomposition of  $x \in \mathcal{S}$ ,  $x = x_0 e_0 + x'$  is explained in (2.2).

Then,  $u_\varepsilon$  converges uniformly to  $u$  in  $L^\infty(\mathcal{S}, \mathbb{R})$  and there exists a function  $m : (0, +\infty) \rightarrow (0, +\infty)$ , such that  $\lim_{\varepsilon \rightarrow 0} m(\varepsilon) = 0$  and the function  $u_\varepsilon - m(\varepsilon)$  is a viscosity subsolution of (2.15) on a neighborhood of  $\Gamma$ .

*Proof.* The uniform convergence of  $u_\varepsilon$  to  $u$  in  $L^\infty(\mathcal{S}, \mathbb{R})$  is classical because  $u$  is bounded and continuous in  $\mathcal{S}$ . The existence of a function  $m : (0, +\infty) \rightarrow (0, +\infty)$  with  $m(0^+) = 0$  such that  $u_\varepsilon - m(\varepsilon)$  is a subsolution of (2.15) on a neighborhood of  $\Gamma$  was proved by P.L. Lions [18] or Barles & Jakobsen [5]. A crucial information for obtaining this result is the fact that, according to Lemma 2.6,  $u$  is Lipschitz continuous on  $B(\Gamma, r) \cap \mathcal{S}$ , for some positive number  $r$ .  $\square$

**Theorem 2.5.** *Assume [H0], [H1], [H2] and [H3]. Let  $u : \mathcal{S} \rightarrow \mathbb{R}$  be a bounded continuous viscosity subsolution of (2.15), and  $v : \mathcal{S} \rightarrow \mathbb{R}$  be a bounded viscosity supersolution of (2.15). Then  $u \leq v$  in  $\mathcal{S}$ .*

*Proof.* The strategy of proof adopted here is the one of Barles-Briani-Chasseigne [6] in the proof of their theorem 4.1. It is adapted to the present context.

Let  $u_\varepsilon$  be the approximation of  $u$  given by Lemma 2.9. It is a simple matter to check that there exists a positive real number  $M$  such that the function  $\psi(x) = -|x|^2 - M$  is a viscosity subsolution of (2.15). For  $0 < \mu < 1$ ,  $\mu$  close to 1, the function  $u_{\varepsilon, \mu} = \mu(u_\varepsilon - m(\varepsilon)) + (1 - \mu)\psi$  is a viscosity subsolution of (2.15), which tends to  $-\infty$  as  $|x|$  tends to  $+\infty$ . Let  $M_{\varepsilon, \mu}$  be the maximal value of  $u_{\varepsilon, \mu} - v$  which is reached at some point  $\bar{x}_{\varepsilon, \mu}$ . We argue by contradiction assuming that  $M_{\varepsilon, \mu} > 0$ .

1. If  $\bar{x}_{\varepsilon, \mu} \notin \Gamma$ , then we introduce the function  $u_{\varepsilon, \mu}(x) - v(x) - d^2(x, \bar{x}_{\varepsilon, \mu})$ , where  $d(\cdot, \cdot)$  is the geodesic distance defined by (2.3), which has a strict maximum at  $\bar{x}_{\varepsilon, \mu}$ , and we double the variables, i.e. for  $0 < \beta \ll 1$ , we consider

$$(x, y) \mapsto u_{\varepsilon, \mu}(x) - v(y) - d^2(x, \bar{x}_{\varepsilon, \mu}) - \frac{d^2(x, y)}{\beta^2}.$$

Classical arguments then lead to the conclusion that  $u_{\varepsilon, \mu}(\bar{x}_{\varepsilon, \mu}) - v(\bar{x}_{\varepsilon, \mu}) \leq 0$ , thus  $M_{\varepsilon, \mu} \leq 0$ , which is a contradiction.

2. If  $\bar{x}_{\varepsilon, \mu} \in \Gamma$ , according to Lemma 2.6,  $u_{\varepsilon, \mu}$  is Lipschitz continuous in a neighborhood of  $\Gamma$ . Then, with a similar argument as in the proof of Lemma 2.7, we can construct a test function  $\varphi \in \mathcal{R}(\mathcal{S})$  such that  $\varphi|_\Gamma = u_{\varepsilon, \mu}|_\Gamma$  and  $\varphi$  remains below  $u_{\varepsilon, \mu}$  on a neighborhood of  $\Gamma$  (take for exemple  $\varphi(x) = u_{\varepsilon, \mu}(x_0, 0) - Cx_i$  if  $x = x_0e_0 + x_ie_i \in \mathcal{P}_i$ , with  $C$  great enough). It is easy to check that  $v - \varphi$  has a local minimum at  $\bar{x}_{\varepsilon, \mu}$ , from the assumption that  $M_{\varepsilon, \mu} > 0$ . Then, we can use Theorem 2.4 and we have two possible cases:

[B]  $\lambda v(\bar{x}_{\varepsilon, \mu}) + H_\Gamma^T(\bar{x}_{\varepsilon, \mu}, D(u_{\varepsilon, \mu}|_\Gamma)(\bar{x}_{\varepsilon, \mu})) \geq 0$ .

Moreover,  $u_{\varepsilon, \mu}$  is a subsolution of (2.15),  $C^1$  on  $\Gamma$ . Then, according to Lemma 2.7, we have the inequality  $\lambda u_{\varepsilon, \mu}(\bar{x}_{\varepsilon, \mu}) + H_\Gamma^T(\bar{x}_{\varepsilon, \mu}, D(u_{\varepsilon, \mu}|_\Gamma)(\bar{x}_{\varepsilon, \mu})) \leq 0$ . Therefore, we obtain that  $u_{\varepsilon, \mu}(\bar{x}_{\varepsilon, \mu}) \leq v(\bar{x}_{\varepsilon, \mu})$ , thus  $M_{\varepsilon, \mu} \leq 0$ , which is a contradiction.

[A] With the notations of Theorem 2.4, we have that

$$v(x_k) \geq \int_0^\eta \ell_i(y_{x_k}(s), \alpha_i^k(s)) e^{-\lambda s} ds + v(y_{x_k}(\eta)) e^{-\lambda \eta}.$$

Moreover, from Lemma 2.8,

$$u_{\varepsilon, \mu}(x_k) \leq \int_0^\eta \ell_i(y_{x_k}(s), \alpha_i^k(s)) e^{-\lambda s} ds + u_{\varepsilon, \mu}(y_{x_k}(\eta)) e^{-\lambda \eta}.$$

Therefore

$$u_{\varepsilon, \mu}(x_k) - v(x_k) \leq (u_{\varepsilon, \mu}(y_{x_k}(\eta)) - v(y_{x_k}(\eta))) e^{-\lambda \eta}.$$

Letting  $k$  tend to  $+\infty$ , we find that  $M_{\varepsilon, \mu} \leq M_{\varepsilon, \mu} e^{-\lambda \eta}$ , which implies that  $M_{\varepsilon, \mu} \leq 0$ , which is a contradiction.

We conclude by letting  $\varepsilon$  tend to 0 and  $\mu$  tend to 1.  $\square$

**Theorem 2.6.** *Assume [H0], [H1], [H2] and [H3]. Then, the value function  $v$  is the unique viscosity solution of (2.15) in  $\mathcal{S}$ .*

**Remark 2.11.** Under the assumptions [H0], [H1], [H2] and [H3] it is possible to prove a more general Comparison principle, where we do not assume the continuity of the subsolutions. The prove of a such Comparison principle is slightly more technical. Using that any subsolution of (2.15) is Lipschitz continuous in a neighborhood of  $\Gamma$ , from Lemma 2.6, we are first able to prove a local Comparison principle similar to Theorem 3.3 and then to deduce a global Comparison principle similar to Theorem 3.4. Essentially, we have to follow the proofs of Theorem 3.3 and Theorem 3.4, with removing of the step of regularisation by sup-convolution.

### 3 Second case : normal controllability near interface

#### 3.1 The new framework

We keep assumptions [H0], [H1], [H2] and we weaken the controllability assumption [H3] by only supposing normal controllability,

[ $\tilde{H}3$ ] There is a real number  $\delta > 0$  such that for any  $i = 1, \dots, N$  and for all  $x \in \Gamma$ ,

$$[-\delta, \delta] \subset \{f_i(x, a).e_i : a \in A_i\}.$$

The following property is the counterpart of property 2.2 in the current framework.

**Property 3.1.** Assume [H0] and [ $\tilde{H}3$ ]. Then, there exists  $R > 0$  a positive real number such that for all  $i = 1, \dots, N$  and  $x \in B(\Gamma, R) \cap \mathcal{P}_i$

$$[-\frac{\delta}{2}, \frac{\delta}{2}] \subset \{f_i(x, a).e_i : a \in A_i\}. \quad (3.1)$$

The dynamics  $f$  and the cost function  $\ell$  are defined on

$$M = \{(x, a); x \in \mathcal{S}, a \in A_i \text{ if } x \in \mathcal{P}_i \setminus \Gamma, \text{ and } a \in \cup_{i=1}^N A_i \text{ if } x \in \Gamma\},$$

as in (2.5) and (2.9). As above, we need to introduce the set of the admissible controlled trajectories. For this purpose, we recall that for  $x \in \mathcal{S}$ ,

$$\begin{aligned} \tilde{F}(x) &= \begin{cases} F_i(x) & \text{if } x \text{ belongs to the open half-plane } \mathcal{P}_i \setminus \Gamma \\ \cup_{i=1}^N F_i(x) & \text{if } x \in \Gamma, \end{cases} \\ Y_x &= \left\{ y_x \in \text{Lip}(\mathbb{R}^+; \mathcal{S}) : \begin{cases} \dot{y}_x(t) \in \tilde{F}(y_x(t)), & \text{for a.a. } t > 0 \\ y_x(0) = x, \end{cases} \right\}. \end{aligned} \quad (3.2)$$

**Theorem 3.1.** Assume [H0], [H2] and [ $\tilde{H}3$ ]. Then

1. For any  $x \in \mathcal{S}$ ,  $Y_x$  is not empty.
2. For any  $x \in \mathcal{S}$ , for each trajectory  $y_x \in Y_x$ , there exists a measurable function  $\Phi : [0, +\infty) \rightarrow M$ ,  $\Phi(t) = (\varphi_1(t), \varphi_2(t))$  with

$$(y_x(t), \dot{y}_x(t)) = (\varphi_1(t), f(\varphi_1(t), \varphi_2(t))), \quad \text{for a.a. } t.$$

3. Almost everywhere on  $\{t : y_x(t) \in \Gamma\}$ ,  $f(y_x(t), \varphi_2(t)) \in \mathbb{R}e_0$ .

Points 2 and 3 are proved exactly as in the Theorem 2.2. Point 1 is little bit more difficult to prove because with [ $\tilde{H}3$ ], it may happen that  $0 \notin \tilde{F}(x)$  when  $x \in \Gamma$ . Here, point 1 is essentially a consequence of the following lemma.

**Lemma 3.1.** Assume [H0], [H2] and [ $\tilde{H}3$ ]. There exists  $T > 0$  such that  $\forall i \in \{1, \dots, N\}$  and  $\forall x \in \mathcal{P}_i$ , there exists  $y_x \in \text{Lip}([0, T]; \mathcal{P}_i)$  a solution of the differential inclusion

$$\begin{cases} \dot{y}_x(t) \in \tilde{F}(y_x(t)) \\ y_x(0) = x. \end{cases}$$



*Proof.* Take  $x \in \mathcal{P}_i$ .

- **If  $x \in \Gamma$  :** according to [H3], there exists  $a_i \in A_i$  such that  $f_i(x, a_i) \cdot e_i = \delta$ . Then, from the Lipschitz property in [H0], we have

$$f_i(z, a_i) \cdot e_i \geq \frac{\delta}{2}, \quad \forall z \in B(x, \frac{\delta}{2L_f}) \cap \mathcal{P}_i. \quad (3.3)$$

As a consequence, if we set  $T = \frac{\delta}{2L_f M_f}$ , there exists a unique function  $y_x : [0, T] \rightarrow \mathcal{P}_i$  such that  $y_x(t) = x + \int_0^t f_i(y_x(s), a_i) ds$  and  $y_x(t) \in B(x, \frac{\delta}{2L_f}) \cap \mathcal{P}_i \forall t \in (0, T]$ .

- **If  $x \in \mathcal{P}_i \setminus \Gamma$  :** then, we chose arbitrarily  $a_i \in A_i$  and we consider  $\bar{y} : [0, \bar{T}] \rightarrow \mathcal{P}_i$  the maximal solution of the integral equation  $y(t) = x + \int_0^t f_i(y(s), a_i) ds$ . If  $\bar{T} > T = \frac{\delta}{2L_f M_f}$  then we take  $y_x = \bar{y}$ . Otherwise  $\bar{y}(\bar{T})$  is well defined and belongs to  $\Gamma$ , then we are reduced to the case where  $x \in \Gamma$ .

□

Now, we are able to prove Point 1 of Theorem 3.1.

*Proof.* Let  $x$  be in  $\mathcal{P}_i$ , for some  $i \in \{1, \dots, N\}$ . According to Lemma 3.1 we are able to build a sequence of positive reals  $(T_n)_{n \in \mathbb{N}}$  and maps  $y_n : [0, T_n] \rightarrow \mathcal{P}_i$  such that, for any  $n \in \mathbb{N}$

- (i)  $y_n(0) = x$  and  $\dot{y}_n(t) \in F_i(y_n(t))$ , for a.a.  $t \in (0, T_n]$
- (ii)  $y_n(t) \in \mathcal{P}_i, \forall t \in (0, T_n]$
- (iii)  $T_{n+1} - T_n \geq T$ , with  $T$  given by Lemma 3.1
- (iv)  $\forall 0 \leq q \leq p, y_q|_{[0, T_p]} \equiv y_p$

So, the map  $y_x : \mathbb{R}_+ \rightarrow \mathcal{P}_i$  defined by  $y_x(t) = y_n(t)$  if  $t \in [0, T_n]$  belongs to  $Y_x$  and particularly Point 1 of Theorem 3.1 is true. □

So, as above, we can take the set of admissible controlled trajectories starting from the initial datum  $x$ :

$$\mathcal{T}_x = \left\{ (y_x, \alpha) \in L_{\text{Loc}}^\infty(\mathbb{R}^+; M) : \begin{cases} y_x \in \text{Lip}(\mathbb{R}^+; \mathcal{S}), \\ y_x(t) = x + \int_0^t f(y_x(s), \alpha(s)) ds \quad \text{in } \mathbb{R}^+ \end{cases} \right\}.$$

Then, the cost functional  $J$  and the value function  $v$  are defined by

$$J(x; (y_x, \alpha)) = \int_0^\infty \ell(y_x(t), \alpha(t)) e^{-\lambda t} dt,$$

where  $\lambda > 0$ , and

$$v(x) = \inf_{(y_x, \alpha) \in \mathcal{T}_x} J(x; (y_x, \alpha)).$$

Unlike in § 2, we cannot use classical arguments to prove that  $v$  is continuous, because we do not suppose [H3] any longer. The main problem is that with [H3] we are no longer sure that for each  $x, z$  close to  $\Gamma$ , there exists an admissible trajectory  $y_{x,z} \in \mathcal{T}_x$  from  $x$  to  $z$ . We will prove later that  $v$  is continuous, but for the moment  $v$  is a priori a discontinuous function. In order to deal with this a priori discontinuity, we use the following notions :

**Definition 3.1.** Let  $u : \mathcal{S} \rightarrow \mathbb{R}$  be a bounded function.

- The lower semi-continuous envelope of  $u$  is defined by

$$u_\star(x) = \liminf_{z \rightarrow x} u(z).$$

- The upper semi-continuous envelope of  $u$  is defined by

$$u^\star(x) = \limsup_{z \rightarrow x} u(z).$$

### 3.2 Hamilton-Jacobi equation

**Definition 3.2.** A bounded function  $u : \mathcal{S} \rightarrow \mathbb{R}$  is a discontinuous viscosity solution of (2.15) in  $\mathcal{S}$  if  $u^*$  is a subsolution and  $u_*$  is a supersolution of (2.15) in  $\mathcal{S}$ .

The next lemmas will be used to prove the existence result.

**Lemma 3.2.** Assume [H0], [H1], [H2] and  $\tilde{[H3]}$ . There exists some constants  $r > 0$  and  $C > 0$  such that for all  $x = x_0 e_0 + x_i e_i \in B(\Gamma, r) \cap (\mathcal{P}_i \setminus \Gamma)$ , there exists an admissible controlled trajectory  $(y_x, \alpha_x) \in \mathcal{T}_x$  such that  $\tau_x \leq Cx_i$ , where  $\tau_x$  is the exit time of  $y_x$  from  $\mathcal{P}_i \setminus \Gamma$ .

**Lemma 3.3.** Assume [H0], [H1], [H2] and  $\tilde{[H3]}$ . For all  $x \in \Gamma$ ,  $i \in \{1, \dots, N\}$  and  $a \in A_i$  such that  $f_i(x, a).e_i \geq 0$ , there exists a sequence  $a_\varepsilon \in A_i$  such that

$$f_i(x, a_\varepsilon).e_i \geq \delta\varepsilon > 0, \quad (3.4)$$

$$|f_i(x, a_\varepsilon) - f_i(x, a)| \leq 2M_f\varepsilon, \quad (3.5)$$

$$|\ell_i(x, a_\varepsilon) - \ell_i(x, a)| \leq 2M_\ell\varepsilon. \quad (3.6)$$

*Proof.* From  $\tilde{[H3]}$  there exists  $a_\delta \in A_i$  such that  $f_i(x, a_\delta).e_i = \delta$ . From [H2],

$$\varepsilon(f_i(x, a_\delta), \ell_i(x, a_\delta)) + (1 - \varepsilon)(f_i(x, a), \ell_i(x, a)) \in \text{FL}_i(x)$$

for any  $\varepsilon \in [0, 1]$ . So, there exists  $a_\varepsilon \in A_i$  such that

$$\varepsilon(f_i(x, a_\delta), \ell_i(x, a_\delta)) + (1 - \varepsilon)(f_i(x, a), \ell_i(x, a)) = (f_i(x, a_\varepsilon), \ell_i(x, a_\varepsilon))$$

which has all the desired properties.  $\square$

**Corollary 3.1.** For any  $i \in \{1, \dots, N\}$ ,  $x \in \Gamma$  and  $p_i \in \mathbb{R}e_0 \times \mathbb{R}e_i$ ,

$$\max_{a \in A_i \text{ s.t. } f_i(x, a).e_i \geq 0} (-p_i f_i(x, a) - \ell_i(x, a)) = \sup_{a \in A_i \text{ s.t. } f_i(x, a).e_i > 0} (-p_i f_i(x, a) - \ell_i(x, a)). \quad (3.7)$$

**Theorem 3.2.** Assume [H0], [H1], [H2] and  $\tilde{[H3]}$ . The value function  $v$  defined in (2.10) is a bounded discontinuous viscosity solution of (2.15) in  $\mathcal{S}$ .

*Proof.* The proof being a bit long, we postpone it to the appendix B.  $\square$

We give now some results on the behavior of the Hamiltonians in the new framework.

**Lemma 3.4.** Assume [H0], [H1], [H2] and  $\tilde{[H3]}$ . Then, for  $i = 1, \dots, N$ , we have

$$|H_i(x, p) - H_i(y, p)| \leq L_f|x - y| + \omega_\ell(|x - y|), \quad \text{for any } x, y \in \mathcal{P}_i \text{ and } p \in \mathbb{R}e_0 \times \mathbb{R}e_i, \quad (3.8)$$

and

$$|H_i(x, p) - H_i(x, q)| \leq M_f|p - q|, \quad \text{for any } x \in \mathcal{P}_i \text{ and } p, q \in \mathbb{R}e_0 \times \mathbb{R}e_i, \quad (3.9)$$

where  $L_f, M_f$  and  $\omega_\ell$  are defined in [H0] and [H1].

At last, if  $C_M = \max\{M_f, M_l\}$ ,

$$H_i(x, p_0 e_0 + p_i e_i) \geq \frac{\delta}{2}|p_i| - C_M(1 + |p_0|), \quad \text{for any } x \in B(\Gamma, R) \cap \mathcal{P}_i \text{ and } p_0, p_i \in \mathbb{R}, \quad (3.10)$$

where  $R > 0$  is a positive number as in (3.1).

*Proof.* The proof of the Lemma is given in [7] Lemma 7.1 We supply it for completeness. Assumptions [H0] and [H1] yield (3.8) and (3.9). For (3.10) we also have to use the partial controllability assumption  $\tilde{[H3]}$ . Indeed, according to Property 3.1 there exist some controls  $a_1, a_2 \in A_i$  such that

$$-f_i(x, a_1).e_i = \frac{\delta}{2} > 0, \quad -f_i(x, a_2).e_i = -\frac{\delta}{2}.$$

Now we compute  $H_i(x, p_0 e_0 + p_i e_i)$  assuming that  $p_i > 0$  (the other case is treated similarly).

$$\begin{aligned} H_i(x, p_0 e_0 + p_i e_i) &\geq -f_i(x, a_1) \cdot (p_0 e_0 + p_i e_i) - \ell_i(x, a_1) \\ &\geq \frac{\delta}{2} |p_i| - f_i(x, a_1) \cdot p_0 e_0 - \ell_i(x, a_1) \\ &\geq \frac{\delta}{2} |p_i| - C_M |p_0| - C_M, \end{aligned}$$

the last line coming from the boundedness of  $f_i$  and  $\ell_i$ . This concludes the proof.  $\square$

**Lemma 3.5.** Assume [H0], [H1], [H2] and  $[\tilde{H}3]$ . Then, for any  $x, x' \in \Gamma$ ,  $i \in \{1, \dots, N\}$  and  $a \in A_i$  such that  $f_i(x, a) \cdot e_i \geq 0$ , there exists  $a' \in A_i$  such that  $f_i(x', a') \cdot e_i \geq 0$  and

$$|f_i(x, a) - f_i(x', a')| \leq M|x - x'|, \quad (3.11)$$

$$|\ell_i(x, a) - \ell_i(x', a')| \leq \omega(|x - x'|), \quad (3.12)$$

where, if  $M_f, L_f, M_\ell, \delta$  and  $\omega_\ell$  are given by assumptions [H0], [H1] and  $[\tilde{H}3]$ .

$$M = L_f (1 + 2M_f \delta^{-1}) \quad \text{and} \quad \omega(t) = \omega_\ell(t) + 2M_\ell L_f \delta^{-1} t \quad \text{for } t \geq 0.$$

*Proof.* The proof follows the lines of that of [7], Lemma 7.4 : let  $a \in A_i$  be such that  $f_i(x, a) \cdot e_i \geq 0$ . Fix  $x' \in \Gamma$ , we have two possibilities. If  $f_i(x', a) \cdot e_i \geq 0$  the conclusion follows easily because according to [H0] and [H1] we have respectively (3.11) and (3.12) with  $a' = a$ . Otherwise  $f_i(x', a) \cdot e_i < 0$ . By the partial controllability assumption in  $[\tilde{H}3]$  there exists a control  $a_1 \in A_i$  such that  $f_i(x', a_1) \cdot e_i = \delta$ . We then set

$$\bar{\mu} := \frac{\delta}{\delta - f_i(x', a) \cdot e_i}.$$

Since  $\bar{\mu} \in (0, 1)$ , by the convexity assumption in [H2], there exists a control  $a' \in A_i$  such that

$$\bar{\mu} (f_i(x', a), \ell_i(x', a)) + (1 - \bar{\mu}) (f_i(x', a_1), \ell_i(x', a_1)) = (f_i(x', a'), \ell_i(x', a')).$$

By construction  $f_i(x', a') \cdot e_i = 0$  and then, in particular,  $f_i(x', a') \cdot e_i \geq 0$ . Moreover, since

$$(1 - \bar{\mu}) = \frac{-f_i(x', a) \cdot e_i}{\delta - f_i(x', a) \cdot e_i} \leq \frac{f_i(x, a) \cdot e_i - f_i(x', a) \cdot e_i}{\delta - f_i(x', a) \cdot e_i} \leq \frac{L_f |x - x'|}{\delta},$$

we have

$$\begin{aligned} |f_i(x, a) - f_i(x', a')| &\leq |f_i(x, a) - f_i(x', a)| + |f_i(x', a) - f_i(x', a')| \\ &\leq L_f |x - x'| + (1 - \bar{\mu}) |f_i(x', a) - f_i(x', a_1)| \\ &\leq L_f (1 + 2M_f \delta^{-1}) |x - x'|. \end{aligned}$$

This proves (3.11). The same calculation with  $\ell_i$  gives us (3.12).  $\square$

**Remark 3.1.** In Lemma 3.5, if  $x \in \Gamma$  and  $a \in A_i$  are such that  $f_i(x, a) \cdot e_i = 0$ , then we have the stronger result that  $\forall x' \in \Gamma, \exists a' \in A_i$  such that  $f_i(x', a') \cdot e_i = 0$  and the inequalities (3.11) and (3.12) are true. Indeed, if  $f_i(x', a) \cdot e_i \leq 0$  the proof of Lemma 3.5 directly provides a suitable  $a' \in A_i$ . We only need to specify the strategy in the case when  $f_i(x', a) \cdot e_i > 0$ . We consider  $a_2 \in A_i$ , given by  $[\tilde{H}3]$  such that  $f_i(x', a_2) \cdot e_i = -\delta$ . Then, we set  $\tilde{\mu} = \frac{\delta}{f_i(x', a) \cdot e_i + \delta} \in (0, 1)$  and we choose  $a' \in A_i$ , given by [H2], such that  $\tilde{\mu} (f_i(x', a), \ell_i(x', a)) + (1 - \tilde{\mu}) (f_i(x', a_2), \ell_i(x', a_2)) = (f_i(x', a'), \ell_i(x', a'))$ . With this choice for  $a' \in A_i$ , the same calculations as in the proof above, using  $1 - \tilde{\mu} = \frac{f_i(x', a) \cdot e_i}{f_i(x', a) \cdot e_i + \delta} = \frac{(f_i(x', a) - f_i(x, a)) \cdot e_i}{f_i(x', a) \cdot e_i + \delta}$ , allow us to conclude.

**Lemma 3.6.** Assume [H0], [H1], [H2] and  $[\tilde{H}3]$ . The Hamiltonian  $H_i^+$  defined in (2.20) has the following properties

$$H_i^+(x, p_0 e_0 + p_i e_i) \geq H_i^+(x, p_0 e_0 + q_i e_i) \quad \forall x \in \Gamma \text{ and } \forall p_0, p_i, q_i \in \mathbb{R} \text{ s.t. } p_i \leq q_i, \quad (3.13)$$

$$H_i^+(x, p_0 e_0 + p_i e_i) \geq -\delta p_i - C_M(1 + |p_0|), \quad \forall x \in \Gamma \text{ and } p_0, p_i \in \mathbb{R}, \quad (3.14)$$

$$|H_i^+(x, p) - H_i^+(y, p)| \leq M|x - y| + \omega(|x - y|) \quad \forall x, y \in \Gamma \text{ and } p \in \mathbb{R} e_0 \times \mathbb{R} e_i, \quad (3.15)$$

where  $M$  and  $\omega$  are defined in Lemma 3.5 and  $C_M = \max\{M_f, M_\ell\}$ .

*Proof.* We skip the easy proof of (3.13). The proof of (3.14) is the same as that of (3.10) with the difference that here the dynamics leading out of  $\mathcal{P}_i$  are not allowed. Let us prove (3.15). Let  $x, y$  be in  $\Gamma$  and  $p$  be in  $\mathbb{R}e_0 \times \mathbb{R}e_i$ . First, there exists  $a \in A_i$  such that  $f_i(x, a) \cdot e_i \geq 0$  and  $H_i^+(x, p) = -f_i(x, a) \cdot p - \ell_i(x, a)$ . Then, we consider  $a' \in A_i$  given by Lemma 3.5 such that  $f_i(y, a') \cdot e_i \geq 0$  and the inequality (3.11) and (3.12) are satisfied. Therefore,

$$\begin{aligned} H_i^+(x, p) - H_i^+(y, p) &\leq (-f_i(x, a) \cdot p - \ell_i(x, a)) - (-f_i(y, a') \cdot p - \ell_i(y, a')) \\ &= (f_i(y, a') - f_i(x, a)) \cdot p + (\ell_i(y, a') - \ell_i(x, a)) \\ &\leq M|x - y||p| + \omega(|x - y|). \end{aligned}$$

We conclude the proof by exchanging the roles of  $x$  and  $y$ .  $\square$

**Remark 3.2.** In view of the calculations above, if the maximum which defines  $H_i^+(x, p_0e_0 + p_ie_i)$  is reached for some  $a \in A_i$  such that  $f_i(x, a) \cdot e_i = 0$ , then from the Remark 3.1 we have the stronger inequality,

$$H_i^+(x, p_0e_0 + p_ie_i) - H_i^+(y, p_0e_0 + p_ie_i) \leq M|x - y||p_0| + \omega(|x - y|), \quad (3.16)$$

but without the modulus on the left side of the inequality.

### 3.3 Comparison principle and Uniqueness

A key argument in the proof of the Comparison Principle in Theorem 2.5 is the fact that the subsolutions of (2.15) are Lipschitz continuous in a neighborhood of  $\Gamma$ . We have not this property in the current framework the above method cannot be applied directly. We will first prove a local comparison principle by reducing ourselves to the case when a subsolution is Lipschitz continuous, and then we will deduce a global comparison principle. For this purpose, we start by stating some useful lemmas.

The following transformation will allow us to focus on the case when the subsolutions are locally Lipschitz continuous in a neighborhood of  $\Gamma$ .

**Definition 3.3.** Let  $u : \mathcal{S} \rightarrow \mathbb{R}$  be a bounded, usc function and  $\alpha$  be a positive number. We define the sup-convolution of  $u$  with respect to the  $x_0$ -variable by

$$u_\alpha(x) := \max_{z_0 \in \mathbb{R}} \left\{ u(z_0e_0 + x_ie_i) - \left( \frac{|z_0 - x_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} \right\}, \quad \text{if } x = x_0e_0 + x_ie_i \in \mathcal{P}_i, \quad (3.17)$$

where  $\alpha, p > 0$  are positive numbers.

We recall well known results on sup-convolution.

**Lemma 3.7.** Let  $u : \mathcal{S} \rightarrow \mathbb{R}$  be a bounded function and  $\alpha, p$  be positive numbers. Then, for any  $x \in \mathcal{S}$  the supremum which defines  $u_\alpha(x)$  is achieved at a point  $z_0 \in \mathbb{R}$  such that

$$\left( \frac{|z_0 - x_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} \leq 2 \|u\|_\infty + \alpha^{\frac{p}{2}}. \quad (3.18)$$

**Lemma 3.8.** Let  $u : \mathcal{S} \rightarrow \mathbb{R}$  be a bounded, usc function. Then, for all  $\alpha, p > 0$ , the sup-convolution  $u_\alpha$  is locally Lipschitz continuous with respect to  $x_0$ ; i.e. for any compact subset  $K \subset \mathbb{R}^3$ , there exists  $C_K > 0$  such that for all  $x = x_0e_0 + x_ie_i, y = y_0e_0 + y_ie_i \in K \cap \mathcal{S}$ ,  $|u_\alpha(x) - u_\alpha(y)| \leq C_K|x_0 - y_0|$ .

**Lemma 3.9.** Assume [H0], [H1], [H2] and  $\widetilde{[H3]}$ . Let  $R > 0$  be a positive number as in (3.1). Let  $u : \mathcal{S} \rightarrow \mathbb{R}$  be a bounded, usc subsolution of (2.15) in  $\mathcal{S}$  and  $\alpha, p$  be some positive numbers. Then, for all  $M > 0$ ,  $u_\alpha$  is Lipschitz continuous in  $B_M(\Gamma, R) \cap \mathcal{S}$ , where  $B_M(\Gamma, R) := \{x = x_0e_0 + x' \in B(\Gamma, R) : |x_0| \leq M\}$ .

*Proof.* Fix  $M > 0$ . To get that  $u_\alpha$  is Lipschitz continuous in  $B_M(\Gamma, R) \cap \mathcal{S}$ , it is enough to show that there exists a positive number  $C^*(M, \alpha, p)$ , which can depend to  $M, \alpha$  and  $p$ , such that  $u_\alpha$  is a subsolution of

$$|Du(x)| \leq C^*(M, \alpha, p) \quad \text{in } B_M(\Gamma, R) \cap \mathcal{S}, \quad (3.19)$$

i.e. for any  $x \in B_M(\Gamma, R)$  and  $\varphi \in \mathcal{R}(\mathcal{S})$ , such that  $u_\alpha - \varphi$  has a local maximum point at  $x$ ,

$$|D(\varphi|_{\mathcal{P}_i})(x)| \leq C^*(M, \alpha, p) \quad \text{if } x \in (B_M(\Gamma, R) \cap \mathcal{P}_i) \setminus \Gamma, \quad (3.20)$$

$$\max_{i=1, \dots, N} \{|\partial_{x_0} \varphi(x)| + [\partial_{x_i}(\varphi|_{\mathcal{P}_i})(x)]_-\} \leq C^*(M, \alpha, p) \quad \text{if } x \in B_M(\Gamma, R) \cap \Gamma, \quad (3.21)$$

where for  $x \in \mathbb{R}$ ,  $[x]_- = \max\{0, -x\}$ .

Indeed, if (3.19) is proved, then the method used in the proof of Lemma 2.6 allows us to get that  $u_\alpha$  is Lipschitz continuous with Lipschitz constant  $C^*(M, \alpha, p)$ .

Let us prove the existence of the constant  $C^*(M, \alpha, p)$ .

According to Lemma 3.8, there exists a constant  $C(M, \alpha, p)$  such that  $u_\alpha$  is  $C(M, \alpha, p)$ -Lipschitz continuous with respect to the  $x_0$ -variable. A direct consequence of this is that  $u_\alpha$  is a subsolution of

$$|\partial_{x_0} u(x)| \leq C(M, \alpha, p) \quad \text{in } B_M(\Gamma, R) \cap \mathcal{S}. \quad (3.22)$$

i.e. for any  $x \in B_M(\Gamma, R) \cap \mathcal{S}$  and  $\varphi \in \mathcal{R}(\mathcal{S})$ , such that  $u_\alpha - \varphi$  has a local maximum point at  $x$

$$|\partial_{x_0} \varphi(x)| \leq C(M, \alpha, p).$$

It remains to get information on  $|\partial_{x_i}(u_\alpha|_{\mathcal{P}_i})|$  in a viscosity sense. Let  $\bar{x} = \bar{x}_0 e_0 + \bar{x}_i e_i \in B_M(\Gamma, R) \cap \mathcal{P}_i$  and  $\varphi \in \mathcal{R}(\mathcal{S})$  be such that  $u_\alpha - \varphi$  has a local maximum at  $\bar{x}$ . According to Lemma 3.7, there exists  $\bar{z}_0 \in \mathbb{R}$  such that

$$u_\alpha(\bar{x}) := u(\bar{z}_0 e_0 + \bar{x}_i e_i) - \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} \quad (3.23)$$

and

$$\left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} \leq 2 \|u\|_\infty + \alpha^{\frac{p}{2}}. \quad (3.24)$$

Then, if we set  $\varphi_\alpha : z_0 e_0 + x_i e_i \mapsto \left( \frac{|z_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} + \varphi(\bar{x}_0 e_0 + x_i e_i)$ , we can check that  $\varphi_\alpha$  belongs to  $\mathcal{R}(\mathcal{S})$  and that  $u - \varphi_\alpha$  has a local maximum at  $\bar{z} = \bar{z}_0 e_0 + \bar{x}_i e_i$ . Since  $u$  is a subsolution of (2.15) in  $\mathcal{S}$ , we deduce that

$$\lambda u(\bar{z}) + \sup_{(\zeta, \xi) \in \text{FL}(\bar{z})} \{-D\varphi_\alpha(\bar{z}, \zeta) - \xi\} \leq 0. \quad (3.25)$$

**If  $\bar{x} \in (B_M(\Gamma, R) \cap \mathcal{P}_i) \setminus \Gamma$ :** The equation (3.25) can be rewritten as follows

$$\lambda u(\bar{z}) + H_i(\bar{z}_0 e_0 + \bar{x}_i e_i, \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{x}_0 - \bar{z}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + \partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x}_0 e_0 + \bar{x}_i e_i) e_i \leq 0. \quad (3.26)$$

Then, according to (3.10) in Lemma 3.4

$$\lambda u(\bar{z}) + \frac{\delta}{2} |\partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x})| - C_M \left( 1 + \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{x}_0 - \bar{z}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} \right) \leq 0. \quad (3.27)$$

But from (3.24), the term  $\frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{x}_0 - \bar{z}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1}$  is bounded by a constant  $K(\alpha, p) = \frac{p\sqrt{(2\|u\|_\infty + \alpha^{\frac{p}{2}})^{\frac{2}{p}-\alpha}}}{\alpha} (2\|u\|_\infty + \alpha^{\frac{p}{2}})^{1-\frac{2}{p}}$ , so that we get

$$|\partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x})| \leq \frac{2}{\delta} (\lambda \|u\|_\infty + C_M (1 + K(M, \alpha, p))). \quad (3.28)$$

If  $\bar{x} \in B_M(\Gamma, R) \cap \Gamma$ : equation (3.25) can be rewritten as follows

$$\begin{aligned} \lambda u(\bar{z}) + H_\Gamma \left( \bar{z}, \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{x}_0 - \bar{z}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + \partial_{x_1}(\varphi|_{\mathcal{P}_1})(\bar{x})e_1, \dots \right. \\ \left. \dots, \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{x}_0 - \bar{z}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + \partial_{x_N}(\varphi|_{\mathcal{P}_N})(\bar{x})e_N \right) \leq 0. \end{aligned} \quad (3.29)$$

Particularly, if we fix  $i \in \{1, \dots, N\}$ , (3.29) implies

$$\lambda u(\bar{z}) + H_i^+ \left( \bar{z}, \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{x}_0 - \bar{z}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + \partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x})e_i \right) \leq 0, \quad (3.30)$$

and thanks to (3.14) in Lemma 3.6 and (3.24) we deduce that

$$-\partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x}) \leq \frac{1}{\delta} (\lambda \|u\|_\infty + C_M(1 + K(M, \alpha, p))), \quad (3.31)$$

with the same constant  $K(M, \alpha, p)$  as in the case where  $\bar{x} \in B_M(\Gamma, R) \cap \mathcal{P}_i$ .

Finally, according to (3.22), (3.28) and (3.31) we have that  $u_\alpha$  is a subsolution of (3.19) in

$B_M(\Gamma, R) \cap \mathcal{S}$  with the constant  $C^*(M, \alpha, p) = \sqrt{C(M, \alpha, p)^2 + \frac{4}{\delta^2} (\lambda \|u\|_\infty + C_M(1 + K(M, \alpha, p)))^2}$ . This concludes the proof.  $\square$

**Lemma 3.10.** Assume [H0], [H1], [H2],  $\widetilde{[H3]}$  and [H4]. Let  $y_0$  be in  $\Gamma$  and  $R > 0$  be as in (3.1). We denote by  $Q$  the set  $\mathcal{S} \cap B(y_0, R)$ . Let  $u : \mathcal{S} \rightarrow \mathbb{R}$  be a bounded, usc subsolution of (2.15) in  $Q$ . Then, for all  $\alpha, p > 0$  small enough, if we set

$$Q_\alpha := \left\{ x \in Q : \text{dist}(x, \partial Q) > \alpha \sqrt{(2 \|u\|_\infty + \alpha^{\frac{p}{2}})^{2/p} - \alpha} \right\}, \quad (3.32)$$

the sup-convolution  $u_\alpha$  defined in (3.17), is Lipschitz continuous in  $Q_\alpha$  and there exists  $m : (0, +\infty) \rightarrow (0, +\infty)$  such that  $\lim_{\alpha \rightarrow 0} m(\alpha) = 0$  and  $u_\alpha - m(\alpha)$  is a subsolution of (2.15) in  $Q_\alpha$ .

*Proof.* First, according to Lemma 3.9 it is clear that  $u_\alpha$  is Lipschitz continuous in  $Q_\alpha$ . It remains to find  $m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $u_\alpha - m(\alpha)$  is a subsolution of (2.15) in  $Q_\alpha$  and to check that  $m$  has the desired limit as  $\alpha$  tends to 0. For this purpose, we consider  $\varphi \in \mathcal{R}(\mathcal{S})$  such that  $u_\alpha - \varphi$  has a local maximum point at  $\bar{x} = \bar{z}_0 e_0 + \bar{x}_i e_i$ . According to Lemma 3.7, there exists  $\bar{z}_0 \in \mathbb{R}$  such that

$$u_\alpha(\bar{x}) = u(\bar{z}_0 e_0 + \bar{x}_i e_i) - \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} \quad (3.33)$$

and

$$|\bar{x}_0 - \bar{z}_0| \leq \alpha \sqrt{(2 \|u\|_\infty + \alpha^{\frac{p}{2}})^{\frac{2}{p}} - \alpha}. \quad (3.34)$$

From (3.33) the function  $x_0 \mapsto u(\bar{z}_0 e_0 + \bar{x}_i e_i) - \left( \frac{|\bar{z}_0 - x_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} - \varphi(x_0 e_0 + \bar{x}_i e_i)$  has a local maximum at  $\bar{x}_0$  and we deduce that

$$\partial_{x_0} \varphi(\bar{x}) = \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1}. \quad (3.35)$$

On the other hand, since  $\bar{x} \in Q_\alpha$ , the inequality (3.34) implies that  $\bar{z} = \bar{z}_0 e_0 + \bar{x}_i e_i \in Q$  and we can use that  $u$  is a subsolution of (2.15) in  $Q$  :

1. If  $\bar{x} \in \mathcal{P}_i \setminus \Gamma$ : then,  $\bar{z} = \bar{z}_0 e_0 + \bar{x}_i e_i$  also belongs to  $\mathcal{P}_i \setminus \Gamma$  and using the test function

$\tilde{\varphi} : z = z_0 e_0 + z' \mapsto \varphi(\bar{x}_0 e_0 + z') + \left( \frac{|z_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}}$ , where we recall that  $z'$  denotes  $z_j e_j$  if  $z \in \mathcal{P}_j$ , we have the viscosity inequality

$$\lambda u(\bar{z}) + H_i \left( \bar{z}, \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + \partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x}) e_i \right) \leq 0. \quad (3.36)$$

Combining the previous results, (3.33), (3.35) and (3.36), we get

$$\lambda u_\alpha(\bar{x}) + \lambda \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} + H_i(\bar{z}, D(\varphi|_{\mathcal{P}_i})(\bar{x})) \leq 0.$$

Then, according to (3.8) we have

$$\lambda u_\alpha(\bar{x}) + H_i(\bar{x}, D(\varphi|_{\mathcal{P}_i})(\bar{x})) \leq -\lambda \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} + L_f |\bar{x}_0 - \bar{z}_0| |D(\varphi|_{\mathcal{P}_i})(\bar{x})| + \omega_\ell(|\bar{x}_0 - \bar{z}_0|). \quad (3.37)$$

But, from (3.10) and (3.36)

$$|\partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x})| \leq \frac{2}{\delta} \left( \lambda \|u\|_\infty + C_M \left( 1 + \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} \right) \right).$$

Then, from (3.35) and (3.37) we get

$$\begin{aligned} \lambda u_\alpha(\bar{x}) + H_i(\bar{x}, D(\varphi|_{\mathcal{P}_i})(\bar{x})) &\leq \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} (pL_f (1 + \frac{2C_M}{\delta}) - \lambda) \\ &\quad + |\bar{x}_0 - \bar{z}_0| \frac{2L_f}{\delta} (\lambda \|u\|_\infty + C_M) + \omega_\ell(|\bar{x}_0 - \bar{z}_0|), \end{aligned}$$

and for  $p$  small enough, we get

$$\lambda u_\alpha(\bar{x}) + H_i(\bar{x}, D(\varphi|_{\mathcal{P}_i})(\bar{x})) \leq |\bar{x}_0 - \bar{z}_0| \frac{2L_f}{\delta} (\lambda \|u\|_\infty + C_M) + \omega_\ell(|\bar{x}_0 - \bar{z}_0|).$$

Finally, according to (3.34) the right hand side of the latter inequality gives us  $m(\alpha)$ .

**2.** If  $\bar{x} \in \Gamma$ :  $\bar{x}_i = 0$  and  $\bar{z} = \bar{z}_0 e_0 + \bar{x}_i e_i$  also belongs to  $\Gamma$ . Using  $\tilde{\varphi} : z = z_0 e_0 + z' \mapsto \varphi(\bar{x}_0 e_0 + z') + \left( \frac{|z_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}}$  as a test function, we have

$$\begin{aligned} \lambda u(\bar{z}) + H_\Gamma \left( \bar{z}, \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + \partial_{x_1}(\varphi|_{\mathcal{P}_1})(\bar{x}) e_1, \dots \right. \\ \left. \dots, \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + \partial_{x_N}(\varphi|_{\mathcal{P}_N})(\bar{x}) e_N \right) \leq 0. \end{aligned} \quad (3.38)$$

As in the case when  $\bar{x} \in \mathcal{P}_i \setminus \Gamma$ , we deduce from (3.33), (3.35) and (3.38) that we have

$$\lambda u_\alpha(\bar{x}) + \lambda \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} + H_\Gamma(\bar{z}, D(\varphi|_{\mathcal{P}_1})(\bar{x}), \dots, D(\varphi|_{\mathcal{P}_N})(\bar{x})) \leq 0. \quad (3.39)$$

Let  $i \in \{1, \dots, N\}$  be such that  $H_\Gamma(\bar{x}, D(\varphi|_{\mathcal{P}_1})(\bar{x}), \dots, D(\varphi|_{\mathcal{P}_N})(\bar{x})) = H_i^+(\bar{x}, D(\varphi|_{\mathcal{P}_i})(\bar{x}))$ . Then, Lemma 3.6 and (3.39) give us

$$\begin{aligned} \lambda u_\alpha(\bar{x}) + H_\Gamma(\bar{x}, D(\varphi|_{\mathcal{P}_1})(\bar{x}), \dots, D(\varphi|_{\mathcal{P}_N})(\bar{x})) &\leq -\lambda \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} + M |\bar{x}_0 - \bar{z}_0| |D(\varphi|_{\mathcal{P}_i})(\bar{x})| \\ &\quad + \omega(|\bar{x}_0 - \bar{z}_0|), \end{aligned} \quad (3.40)$$

where  $M$  and  $\omega(\cdot)$  are specified in Lemma 3.6. On the other hand, from (3.14) in Lemma 3.6 and (3.38)

$$-\partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x}) \leq \frac{1}{\delta} \left( \lambda \|u\|_\infty + C_M \left( 1 + \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} \right) \right). \quad (3.41)$$

This information does not allow us to conclude directly with (3.40). We need to get a control from above on  $\partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x})$ . For this purpose, we introduce the real number  $\tilde{p}_{i,\bar{x},\alpha}$  defined as follow

$$\tilde{p}_{i,\bar{x},\alpha} := \max\{p_i \in \mathbb{R} : \varphi_{i,\bar{x},\alpha}(p_i) = \min_{d \in \mathbb{R}}\{\varphi_{i,\bar{x},\alpha}(d)\}\} \quad (3.42)$$

where

$$\varphi_{i,\bar{x},\alpha}(d) = H_i(\bar{x}, \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + de_i).$$

We claim that if we note  $h_{i,\bar{x}} := \min_{d \in \mathbb{R}}\{H_i(\bar{x}, de_i)\}$ , we have the following estimation

$$\tilde{p}_{i,\bar{x},\alpha} \leq \frac{2}{\delta} \left( h_{i,\bar{x}} + C_M \left( 1 + \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} \right) + M_f \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} \right). \quad (3.43)$$

To prove this inequality, it is enough to show that for any real number  $q$  larger than the right member of (3.43),  $\varphi_{i,\bar{x},\alpha}(q) > \min_{d \in \mathbb{R}}\{\varphi_{i,\bar{x},\alpha}(d)\}$ .

Take  $q > \frac{2}{\delta} \left( h_{i,\bar{x}} + C_M \left( 1 + \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} \right) + M_f \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} \right)$ .

Then, from the coercivity property of  $H_i$  with respect to  $p_i$ , see (3.10), we have

$$\begin{aligned} H_i(\bar{x}, \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + qe_i &\geq \frac{\delta}{2}|q| - C_M \left( 1 + \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} \right) \\ &> h_{i,\bar{x}} + M_f \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1}. \end{aligned} \quad (3.44)$$

But, if we consider  $d \in \mathbb{R}$  such that  $H_i(\bar{x}, de_i) = h_{i,\bar{x}}$ , by definition of  $p_{i,\bar{x},\alpha}$  we have

$$H_i(\bar{x}, \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + \tilde{p}_{i,\bar{x},\alpha} e_i \leq H_i(\bar{x}, \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + de_i), \quad (3.45)$$

and from (3.9)

$$H_i(\bar{x}, \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + de_i \leq h_{i,\bar{x}} + M_f \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1}. \quad (3.46)$$

Finally, (3.45) and (3.46) lead to

$$H_i(\bar{x}, \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + p_{i,\bar{x},\alpha} e_i \leq h_{i,\bar{x}} + M_f \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1}, \quad (3.47)$$

and from (3.44) we infer that

$$H_i(\bar{x}, \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + qe_i > H_i(\bar{x}, \frac{p(\bar{z}_0 - \bar{x}_0)}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} e_0 + \tilde{p}_{i,\bar{x},\alpha} e_i,$$

which proves (3.43).

We consider three cases :

(a) If  $\partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x}) \leq 0$ : then, (3.41) yields

$$|\partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x})| \leq \frac{1}{\delta} \left( \lambda \|u\|_{\infty} + C_M \left( 1 + \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} \right) \right),$$

and therefore (3.35) and (3.40) imply

$$\begin{aligned} \lambda u_{\alpha}(\bar{x}) + H_{\Gamma}(\bar{x}, D(\varphi|_{\mathcal{P}_1})(\bar{x}), \dots, D(\varphi|_{\mathcal{P}_N})(\bar{x})) &\leq -\lambda \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} + \omega(|\bar{x}_0 - \bar{z}_0|) \\ &+ |\bar{x}_0 - \bar{z}_0| M \left[ \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} + \frac{1}{\delta} \left( \lambda \|u\|_{\infty} + C_M \left( 1 + \frac{p|\bar{z}_0 - \bar{x}_0|}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} \right) \right) \right], \end{aligned}$$



and then

$$\begin{aligned} \lambda u_\alpha(\bar{x}) + H_\Gamma(\bar{x}, D(\varphi|_{\mathcal{P}_1})(\bar{x}), \dots, D(\varphi|_{\mathcal{P}_N})(\bar{x})) &\leq \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} (-\lambda + pM(1 + \frac{C_M}{\delta})) \\ &\quad + \frac{M}{\delta} |\bar{z}_0 - \bar{x}_0| (\lambda \|u\|_\infty + C_M) + \omega(|\bar{x}_0 - \bar{z}_0|). \end{aligned} \quad (3.48)$$

Finally, if  $p$  is small enough, (3.48) implies

$$\lambda u_\alpha(\bar{x}) + H_\Gamma(\bar{x}, D(\varphi|_{\mathcal{P}_1})(\bar{x}), \dots, D(\varphi|_{\mathcal{P}_N})(\bar{x})) \leq \frac{M}{\delta} |\bar{z}_0 - \bar{x}_0| (\lambda \|u\|_\infty + C_M) + \omega(|\bar{x}_0 - \bar{z}_0|), \quad (3.49)$$

and from (3.34) the right hand side of the latter inequality gives us  $m(\alpha)$ .

(b) If  $0 < \partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x}) \leq \tilde{p}_{i,\bar{x},\alpha}$  (case which never occurs if  $p_{i,\bar{x},\alpha} \leq 0$ ): in this case, (3.35) and (3.40) give us

$$\begin{aligned} \lambda u_\alpha(\bar{x}) + H_\Gamma(\bar{x}, D(\varphi|_{\mathcal{P}_1})(\bar{x}), \dots, D(\varphi|_{\mathcal{P}_N})(\bar{x})) &\leq -\lambda \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} \\ &\quad + M \frac{p|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1} + M|\bar{x}_0 - \bar{z}_0| \tilde{p}_{i,\bar{x},\alpha} + \omega(|\bar{x}_0 - \bar{z}_0|), \end{aligned}$$

and according to (3.43) we get

$$\begin{aligned} \lambda u_\alpha(\bar{x}) + H_\Gamma(\bar{x}, D(\varphi|_{\mathcal{P}_1})(\bar{x}), \dots, D(\varphi|_{\mathcal{P}_N})(\bar{x})) &\leq \frac{2M}{\delta} |\bar{x}_0 - \bar{z}_0| (h_{i,\bar{x}} + C_M) + \omega(|\bar{x}_0 - \bar{z}_0|) \\ &\quad + \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} (-\lambda + pM(1 + \frac{2}{\delta}(C_M + M_f))). \end{aligned} \quad (3.50)$$

Finally, for  $p$  small enough, (3.50) gives us

$$\lambda u_\alpha(\bar{x}) + H_\Gamma(\bar{x}, D(\varphi|_{\mathcal{P}_1})(\bar{x}), \dots, D(\varphi|_{\mathcal{P}_N})(\bar{x})) \leq \frac{2M}{\delta} |\bar{x}_0 - \bar{z}_0| (h_{i,\bar{x}} + C_M) + \omega(|\bar{x}_0 - \bar{z}_0|), \quad (3.51)$$

and since  $h_{i,\bar{x}}$  is uniformly bounded with respect to  $\bar{x} \in Q$ , from (3.34) the right hand side of the latter inequality gives us  $m(\alpha)$ .

(c) If  $0 \leq \max\{0, \tilde{p}_{i,\bar{x},\alpha}\} < \partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x})$ : in this case, we can not conclude with the inequality (3.40). We need to find a more precise estimate.

We recall that  $i \in \{1, \dots, N\}$  is such that  $H_\Gamma(\bar{x}, D(\varphi|_{\mathcal{P}_1})(\bar{x}), \dots, D(\varphi|_{\mathcal{P}_N})(\bar{x})) = H_i^+(\bar{x}, D(\varphi|_{\mathcal{P}_i})(\bar{x}))$ . Since  $\tilde{p}_{i,\bar{x},\alpha} < \partial_{x_i}(\varphi|_{\mathcal{P}_i})(\bar{x})$ , according to point 4 in Lemma 2.1, we know that the maximum which defines  $H_i^+(\bar{x}, D(\varphi|_{\mathcal{P}_i})(\bar{x}))$  is reached for one  $a \in A_i$  such that  $f_i(\bar{x}, a) \cdot e_i = 0$ . Then, we can apply (3.16) in Remark 3.2 and (3.39) implies

$$\begin{aligned} \lambda u_\alpha(\bar{x}) + H_\Gamma(\bar{x}, D(\varphi|_{\mathcal{P}_1})(\bar{x}), \dots, D(\varphi|_{\mathcal{P}_N})(\bar{x})) &\leq -\lambda \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}} + \omega(|\bar{x}_0 - \bar{z}_0|) \\ &\quad + M \frac{p|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} \left( \frac{|\bar{z}_0 - \bar{x}_0|^2}{\alpha^2} + \alpha \right)^{\frac{p}{2}-1}. \end{aligned}$$

Then, for  $p$  small enough, we deduce that

$$\lambda u_\alpha(\bar{x}) + H_\Gamma(\bar{x}, D(\varphi|_{\mathcal{P}_1})(\bar{x}), \dots, D(\varphi|_{\mathcal{P}_N})(\bar{x})) \leq \omega(|\bar{x}_0 - \bar{z}_0|), \quad (3.52)$$

and according to (3.34) the right hand side of the latter inequality gives us  $m(\alpha)$ .  $\square$

Finally, let us state the counterpart of Lemma 2.9 in the present context.

**Lemma 3.11.** *Assume [H0], [H1] and [H2]. For  $y_0 \in \Gamma$  let  $R > 0$  be as in (3.1). We set  $Q = B(y_0, R) \cap \mathcal{S}$ . Let  $u : \mathcal{S} \rightarrow \mathbb{R}$  be a bounded, Lipschitz continuous viscosity subsolution of (2.15) in  $Q$ . Let  $(\rho_\varepsilon)_\varepsilon$  be a sequence of mollifiers defined on  $\mathbb{R}$ . We consider the function  $u_\varepsilon$  defined on  $Q_\varepsilon := \{x \in Q : \text{dist}(x, \partial Q) > \varepsilon\}$  by*

$$u_\varepsilon(x_0 e_0 + x') = u * \rho_\varepsilon(x_0 e_0 + x') = \int_{\mathbb{R}} u((x_0 - \tau)e_0 + x') \rho_\varepsilon(\tau) d\tau.$$

*We recall that the decomposition of  $x \in \mathcal{S}$ ,  $x = x_0 e_0 + x'$ , is explained in (2.2).*

*Then,  $\|u_\varepsilon - u\|_{L^\infty(Q_\varepsilon)}$  tends to 0 as  $\varepsilon$  tends to 0 and there exists a function  $\tilde{m} : (0, +\infty) \rightarrow (0, +\infty)$  such that  $\lim_{\varepsilon \rightarrow 0} \tilde{m}(\varepsilon) = 0$  and the function  $u_\varepsilon - \tilde{m}(\varepsilon)$  is a viscosity subsolution of (2.15) in  $Q_\varepsilon$ .*

*Proof.* The proof is similar to that of Lemma 2.9. The difference is that we assume that  $u$  is Lipschitz continuous in  $Q$ . This is no longer a consequence of the assumption [H3].  $\square$

The following theorem is a local Comparison Principle.

**Theorem 3.3.** *Assume [H0], [H1], [H2] and  $\tilde{H}3$ . Let  $u$  be a bounded, usc subsolution of (2.15) in  $\mathcal{S}$  and  $v$  be a bounded, lsc supersolution of (2.15) in  $\mathcal{S}$ . Let  $R > 0$  be as in (3.1). Let  $y_0 \in \Gamma$  be fixed. Then, if we set  $Q = B(y_0, R) \cap \mathcal{S}$ , we have*

$$\| (u - v)_+ \|_{L^\infty(Q)} \leq \| (u - v)_+ \|_{L^\infty(\partial Q)}. \quad (3.53)$$

*Proof. Step 1 :* By assuming  $\tilde{H}3$  instead of [H3], we lose the Lipschitz continuity of  $u$  in a neighborhood of  $\Gamma$ , which was an important property to prove Theorem 2.5. The first step consists therefore of regularizing the subsolution so that it becomes Lipschitz continuous. Take  $\alpha, p > 0$  two positive numbers and consider  $u_\alpha (= u_{\alpha,p})$  the sup-convolution of  $u$  with respect to the  $x_0$ -variable defined in (3.17). We chose  $\alpha, p$  small enough so that Lemma 3.10 can be applied. Thus, from Lemma 3.10, we know that  $u_\alpha$  is Lipschitz continuous in  $Q_\alpha$  and that there exists  $m : (0, +\infty) \rightarrow (0, +\infty)$  such that  $\lim_{\alpha \rightarrow 0} m(\alpha) = 0$  and  $u_\alpha - m(\alpha)$  is a subsolution of (2.15) in  $Q_\alpha$ . The definition of the set  $Q_\alpha (= Q_{\alpha,p})$  is given in (3.32).

**Step 2 :** We are now able to follow the proof of Theorem 2.5. The next step consists of a second regularization of the subsolution  $u$  which this time produces a  $\mathcal{C}^1$  function in  $\Gamma$ . Let  $Q_{\alpha,\varepsilon}$  be the set defined by

$$Q_{\alpha,\varepsilon} := \left\{ x \in Q : \text{dist}(x, \partial Q) > \alpha \sqrt{(2 \|u\|_\infty + \alpha^{\frac{p}{2}})^{2/p} - \alpha + \varepsilon} \right\}.$$

We consider the function  $u_{\alpha,\varepsilon}$  defined on  $Q_{\alpha,\varepsilon}$  by

$$u_{\alpha,\varepsilon}(x_0 e_0 + x') = u_\alpha * \rho_\varepsilon(x_0 e_0 + x') = \int_{\mathbb{R}} u_\alpha((x_0 - \tau)e_0 + x') \rho_\varepsilon(\tau) d\tau,$$

where  $\rho_\varepsilon$  is a sequence of mollifiers defined on  $\mathbb{R}$ . It is clear that  $u_{\alpha,\varepsilon}$  is a  $\mathcal{C}^1$  function in  $\Gamma \cap Q_{\alpha,\varepsilon}$ . Besides, from Lemma 3.11,  $\|u_{\alpha,\varepsilon} - u_\alpha\|_{L^\infty(Q_{\alpha,\varepsilon})}$  tends to 0 as  $\varepsilon$  tends to 0 and there exists a function  $\tilde{m} : (0, +\infty) \rightarrow (0, +\infty)$ , such that  $\lim_{\varepsilon \rightarrow 0} \tilde{m}(\varepsilon) = 0$  and such that the function  $u_{\alpha,\varepsilon} - m(\alpha) - \tilde{m}(\varepsilon)$  is a viscosity subsolution of (2.15) in  $Q_{\alpha,\varepsilon}$ .

**Step 3 :** Let us prove that

$$\| (u_{\alpha,\varepsilon} - m(\alpha) - \tilde{m}(\varepsilon) - v)_+ \|_{L^\infty(Q_{\alpha,\varepsilon})} \leq \| (u_{\alpha,\varepsilon} - m(\alpha) - \tilde{m}(\varepsilon) - v)_+ \|_{L^\infty(\partial Q_{\alpha,\varepsilon})}, \quad (3.54)$$

for a fixed pair  $(\alpha, \varepsilon)$  of positive numbers.

Let  $M_{\alpha,\varepsilon}$  be the supremum of  $u_{\alpha,\varepsilon} - m(\alpha) - \tilde{m}(\varepsilon) - v$  on  $Q_{\alpha,\varepsilon}$ . The latter is reached for some  $\bar{x}_{\alpha,\varepsilon} \in \bar{Q}_{\alpha,\varepsilon}$ , because the function  $u_{\alpha,\varepsilon} - m(\alpha) - \tilde{m}(\varepsilon) - v$  is usc. If  $M_{\alpha,\varepsilon} \leq 0$ , then we clearly have  $\| (u_{\alpha,\varepsilon} - m(\alpha) - \tilde{m}(\varepsilon) - v)_+ \|_{L^\infty(Q_{\alpha,\varepsilon})} \leq \| (u_{\alpha,\varepsilon} - m(\alpha) - \tilde{m}(\varepsilon) - v)_+ \|_{L^\infty(\partial Q_{\alpha,\varepsilon})}$ . So, we assume that  $M_{\alpha,\varepsilon} > 0$  and we want to show that  $\bar{x}_{\alpha,\varepsilon} \in \partial Q_{\alpha,\varepsilon}$ . Assume by contradiction that  $\bar{x}_{\alpha,\varepsilon} \notin \partial Q_{\alpha,\varepsilon}$ . Then,  $\bar{x}_{\alpha,\varepsilon}$  is a local maximum of  $u_{\alpha,\varepsilon} - m(\alpha) - \tilde{m}(\varepsilon) - v$ .

1. If  $\bar{x}_{\alpha,\varepsilon} \notin \Gamma$  : The usual doubling of variables method, with the auxiliary function  $\psi_\beta(x, y) = u_{\alpha,\varepsilon}(x) - m(\alpha) - \tilde{m}(\varepsilon) - v(y) - d^2(x, \bar{x}_{\alpha,\varepsilon}) - \frac{d^2(x, y)}{\beta^2}$ , leads us to a contradiction.
2. If  $\bar{x}_{\alpha,\varepsilon} \in \Gamma$  : According to Lemma 3.10,  $u_{\alpha,\varepsilon}$  is Lipschitz continuous and  $\mathcal{C}^1$  with respect to  $x_0$  in  $\bar{Q}_{\alpha,\varepsilon}$ . Then, with a similar argument as in the proof of Lemma 2.7 we can construct a test-function  $\varphi \in \mathcal{R}(\mathcal{S})$  such that  $\varphi|_\Gamma = u_{\alpha,\varepsilon}|_\Gamma$  and  $\varphi$  remains below  $u_{\alpha,\varepsilon}$  in a neighborhood of  $\Gamma$  (take for example  $\varphi(x_0 e_0 + x_i e_i) = u_{\alpha,\varepsilon}(x_0 e_0) - C x_i$  with  $C$  great enough). It is easy to check that  $v - \varphi$  has a local minimum at  $\bar{x}_{\alpha,\varepsilon}$ . Then, we can use Theorem 2.4, which holds with  $\tilde{H}3$ , and we have two possible cases:

$$[B] \quad \lambda v(\bar{x}_{\alpha,\varepsilon}) + H_\Gamma^T(\bar{x}_{\alpha,\varepsilon}, D(u_{\alpha,\varepsilon}|_\Gamma)(\bar{x}_{\alpha,\varepsilon})) \geq 0.$$

Moreover,  $u_{\alpha,\varepsilon} - m(\alpha) - \tilde{m}(\varepsilon)$  is a subsolution of (2.15) which is  $\mathcal{C}^1$  on  $\Gamma$ . Then,

according to Lemma 2.7, which can be apply here from Remark 2.8, we have the inequality  $\lambda(u_{\alpha,\varepsilon}(\bar{x}_{\alpha,\varepsilon}) - m(\alpha) - \tilde{m}(\varepsilon)) + H_\Gamma^T(\bar{x}_{\alpha,\varepsilon}, D(u_{\alpha,\varepsilon}|\Gamma)(\bar{x}_{\alpha,\varepsilon})) \leq 0$ . Therefore, we obtain that  $M_{\alpha,\varepsilon} = u_{\alpha,\varepsilon}(\bar{x}_{\alpha,\varepsilon}) - m(\alpha) - \tilde{m}(\varepsilon) - v(\bar{x}_{\alpha,\varepsilon}) \leq 0$ , which is a contradiction.

[A] With the notations of Theorem 2.4, we have that

$$v(x_k) \geq \int_0^\eta \ell_i(y_{x_k}(s), \alpha_i^k(s)) e^{-\lambda s} ds + v(y_{x_k}(\eta)) e^{-\lambda \eta}.$$

Moreover, from Lemma 2.8, which holds with  $[\tilde{H}3]$ ,

$$u_{\alpha,\varepsilon}(x_k) - m(\alpha) - \tilde{m}(\varepsilon) \leq \int_0^\eta \ell_i(y_{x_k}(s), \alpha_i^k(s)) e^{-\lambda s} ds + (u_{\alpha,\varepsilon}(y_{x_k}(\eta)) - m(\alpha) - \tilde{m}(\varepsilon)) e^{-\lambda \eta}.$$

Therefore

$$u_{\alpha,\varepsilon}(x_k) - m(\alpha) - \tilde{m}(\varepsilon) - v(x_k) \leq (u_{\alpha,\varepsilon}(y_{x_k}(\eta)) - m(\alpha) - \tilde{m}(\varepsilon) - v(y_{x_k}(\eta))) e^{-\lambda \eta}.$$

Letting  $k$  tend to  $+\infty$ , we find that  $M_{\alpha,\varepsilon} \leq M_{\alpha,\varepsilon} e^{-\lambda \eta}$ , therefore that  $M_{\alpha,\varepsilon} \leq 0$ , which is a contradiction.

**Step 4 :** In order to prove the final result, we have to pass to the limit as  $\varepsilon$  tends to 0 and then as  $\alpha$  tends to 0. Let  $\alpha > 0$  be fixed. Let  $\varepsilon_0$  be a strictly positive number and  $y$  be in  $Q_{\alpha,\varepsilon_0}$ . Then, for all  $0 < \varepsilon < \varepsilon_0$  we have that

$$(u_{\alpha,\varepsilon}(y) - m(\alpha) - \tilde{m}(\varepsilon) - v(y))_+ \leq \| (u_\alpha - m(\alpha) - \tilde{m}(\varepsilon) - v)_+ \|_{L^\infty(\partial Q_{\alpha,\varepsilon})}. \quad (3.55)$$

However,  $\limsup_{\varepsilon \rightarrow 0} \| (u_{\alpha,\varepsilon} - m(\alpha) - \tilde{m}(\varepsilon) - v)_+ \|_{L^\infty(\partial Q_{\alpha,\varepsilon})} \leq \| (u_\alpha - m(\alpha) - v)_+ \|_{L^\infty(\partial Q_\alpha)}$ . Indeed, the supremum  $\| (u_{\alpha,\varepsilon} - m(\alpha) - \tilde{m}(\varepsilon) - v)_+ \|_{L^\infty(\partial Q_{\alpha,\varepsilon})}$  is reached for some  $x_{\alpha,\varepsilon}$  in  $\partial Q_{\alpha,\varepsilon}$ . Thus, for any subsequence such that  $\| (u_{\alpha,\varepsilon} - m(\alpha) - \tilde{m}(\varepsilon) - v)_+ \|_{L^\infty(\partial Q_{\alpha,\varepsilon})}$  converges to a limit  $\bar{\ell}$  when  $\varepsilon$  tends to 0, we can assume that  $x_{\alpha,\varepsilon}$  converges to some  $\bar{x}_\alpha$  that belongs to  $\partial Q_\alpha$  when  $\varepsilon$  tends to 0. Therefore, since  $\| u_{\alpha,\varepsilon} - u_\alpha \|_{L^\infty(Q_{\alpha,\varepsilon})}$  tends to 0 as  $\varepsilon$  tends to 0, since  $u_\alpha$  is continuous in  $Q_\alpha$  and from the lower-semi-continuity of  $v$ , we have that  $\bar{\ell} \leq (u_\alpha(\bar{x}_\alpha) - m(\alpha) - v(\bar{x}_\alpha))_+ \leq \| (u_\alpha - m(\alpha) - v)_+ \|_{L^\infty(\partial Q_\alpha)}$ . Therefore, by the pointwise convergence of  $u_{\alpha,\varepsilon}$  to  $u_\alpha$ , passing to the limsup as  $\varepsilon$  tends to 0 in (3.55) we deduce

$$(u_\alpha(y) - m(\alpha) - v(y))_+ \leq \| (u_\alpha - m(\alpha) - v)_+ \|_{L^\infty(\partial Q_\alpha)}.$$

The above inequality is true for all  $y \in Q_{\alpha,\varepsilon_0}$ , with  $\varepsilon_0$  arbitrarily chosen, then

$$\| (u_\alpha - m(\alpha) - v)_+ \|_{L^\infty(Q_\alpha)} \leq \| (u_\alpha - m(\alpha) - v)_+ \|_{L^\infty(\partial Q_\alpha)}.$$

We are left with taking the limit as  $\alpha$  tends to 0.

Fix now  $\alpha_0$  and  $y \in Q_{\alpha_0}$ . For all  $0 < \alpha \leq \alpha_0$  we have

$$(u_\alpha(y) - m(\alpha) - v(y))_+ \leq \| (u_\alpha - m(\alpha) - v)_+ \|_{L^\infty(\partial Q_\alpha)}. \quad (3.56)$$

As above, we have that  $\limsup_{\alpha \rightarrow 0} \| (u_\alpha - m(\alpha) - v)_+ \|_{L^\infty(\partial Q_\alpha)} \leq \| (u - v)_+ \|_{L^\infty(\partial Q)}$ . Indeed, the supremum  $\| (u_\alpha - m(\alpha) - v)_+ \|_{L^\infty(\partial Q_\alpha)}$  is reached for some  $x_\alpha$  in  $\partial Q_\alpha$ . Thus, for any subsequence such that  $\| (u_\alpha - m(\alpha) - v)_+ \|_{L^\infty(\partial Q_\alpha)}$  converges to a limit  $\ell$  as  $\alpha$  tends to 0, we can assume that  $x_\alpha$  converges to  $\bar{x}$  which belongs to  $\partial Q$  when  $\varepsilon$  tends to 0. But, from the properties of the sup-convolution, the fact that  $u$  is upper-semi-continuous, continuous with respect to  $x_0$  and the fact that  $v$  is lower-semi-continuous it is easy to check that necessarily  $\ell \leq (u(\bar{x}) - v(\bar{x}))_+ \leq \| (u - v)_+ \|_{L^\infty(\partial Q)}$ . Therefore, by the pointwise convergence of  $u_\alpha$  to  $u$ , passing to the limsup as  $\alpha$  tends to 0 in (3.56) we deduce

$$(u(y) - v(y))_+ \leq \| (u - v)_+ \|_{L^\infty(\partial Q)}, \quad \forall y \in Q_{\alpha_0}.$$

The above inequality is true  $\forall y \in Q_{\alpha_0}$ , with  $\alpha_0$  arbitrarily chosen, then

$$\| (u - v)_+ \|_{L^\infty(Q)} \leq \| (u - v)_+ \|_{L^\infty(\partial Q)}.$$

□

We are now able to prove the following global Comparison Principle.

**Theorem 3.4.** *Assume [H0], [H1], [H2] and  $\tilde{H}3$ . Let  $u$  be a bounded, usc subsolution of (2.15) in  $\mathcal{S}$  and  $v$  be a bounded, lsc supersolution of (2.15) in  $\mathcal{S}$ . Then,  $u \leq v$  in  $\mathcal{S}$ .*

*Proof.* The first step consists in a localization of the problem. For a some positive number  $K$ , we consider the function  $\psi(x) := -K - \sqrt{1 + |x|^2}$ . It is easy to check that for  $K \geq \frac{M_f + M_i + 1}{\lambda}$ ,  $\psi$  satisfies the viscosity inequality

$$\lambda\psi + \sup_{(\xi, \zeta) \in \text{FL}(x)} \{-D\psi(x) \cdot \xi - \zeta\} \leq -1.$$

Then, if we set, for  $\mu \in (0, 1)$ ,  $u_\mu = \mu u + (1 - \mu)\psi$ , by convexity properties, we have that

$$\lambda u_\mu + \sup_{(\xi, \zeta) \in \text{FL}(x)} \{-Du_\mu(x) \cdot \xi - \zeta\} \leq -(1 - \mu),$$

where the above inequality is to be understood in the sense of the viscosity. In particular,  $u_\mu$  is a subsolution of (2.15) in  $\mathcal{S}$ . We set  $M_\mu := \sup_{x \in \mathcal{S}} \{u_\mu(x) - v(x)\}$ . Since  $u_\mu(x)$  is usc and tends to  $-\infty$  as  $|x|$  tends to  $+\infty$  and since  $v$  is bounded, lsc the above supremum is reached at some  $x_\mu \in \mathcal{S}$ . We argue by contradiction, assuming that  $M := \sup_{x \in \mathcal{S}} \{u(x) - v(x)\} > 0$ . Then, since  $M_\mu$  tends to  $M$  as  $\mu$  tends to 1, for  $\mu$  close enough to 1 we have  $M_\mu > 0$ . We fix such a  $\mu$  and we distinguish two cases.

1. If  $x_\mu \in \mathcal{P}_i \setminus \Gamma$  for some  $i \in \{1, \dots, N\}$ , then a classical doubling variables method leads to a contradiction.
2. If  $x_\mu \in \Gamma$ , then we are going to obtain a contradiction from Theorem 3.3. Let  $r > 0$  be small enough such that for all  $i \in \{1, \dots, N\}$  and  $x \in B(\Gamma, r) \cap \mathcal{P}_i$

$$\left[-\frac{\delta}{2}, \frac{\delta}{2}\right] \subset \{f_i(x, a) \cdot e_i : a \in A_i\}.$$

We set  $Q_\mu := B(x_\mu, r) \cap \mathcal{S}$  and we consider the function  $\bar{u}_\mu$  defined in  $\mathcal{S}$  by  $\bar{u}_\mu(x) = u_\mu(x) - |x - x_\mu|^2(1 - \mu)^2$ . It is easy to check that if  $\mu$  is close enough to 1,  $\bar{u}_\mu$  is a subsolution of (2.15) in  $\mathcal{S}$ . Indeed, a direct computation gives

$$\lambda \bar{u}_\mu(x) + \sup_{(\xi, \zeta) \in \text{FL}(x)} \{-D\bar{u}_\mu(x, \xi) - \zeta\} \leq -(1 - \mu) + 2rM_f(1 - \mu)^2,$$

and the right hand side of this inequality is clearly negative if  $\mu \in (0, 1) \cap [1 - \frac{1}{2rM_f}, 1)$ . Then, we apply Theorem 3.3 with  $Q = Q_\mu$  and the pair of sub/supersolution  $(\bar{u}_\mu, v)$  : this leads to

$$M_\mu = u_\mu(x_\mu) - v(x_\mu) = \bar{u}_\mu(x_\mu) - v(x_\mu) \leq \|(\bar{u}_\mu - v)_+\|_{L^\infty(\partial Q_\mu)}. \quad (3.57)$$

However, if  $x \in \partial Q_\mu$

$$\bar{u}_\mu(x) - v(x) = u_\mu(x) - v(x) - r^2(1 - \mu)^2 \leq M_\mu - r^2(1 - \mu)^2 < M_\mu,$$

in contradiction with (3.57).

Finally, we deduce that  $M \leq 0$  and the proof is complete.  $\square$

As a consequence, we have the following result of uniqueness and regularity.

**Theorem 3.5.** *Assume [H0], [H1], [H2] and  $\tilde{H}3$ . Then, the value function  $v$  is continuous and is the unique viscosity solution of (2.15) in  $\mathcal{S}$ .*

*Proof.* It is clear that Theorem 3.4 implies the uniqueness for the Hamilton-Jacobi equation (2.15). We have just to prove that  $v$  is a solution of this equation. But, according to Theorem 3.2,  $v$  is a discontinuous solution of (2.15) in  $\mathcal{S}$ . From Theorem 3.4 applied to the pair of sub/supersolution  $(v^*, v_*)$ , we deduce  $v^* \leq v_*$  in  $\mathcal{S}$ . Finally,  $v = v^* = v_*$  and  $v$  is continuous.  $\square$

## 4 Extension to a more general framework with additional dynamics and cost at the interface

With either [H3] or  $[\tilde{H}3]$ , it is possible to extend all the results presented above to the case when there are additional dynamics and cost at the interface. Since the framework is more general with  $[\tilde{H}3]$ , we only discuss this case. We keep the setting from § 3 except that we take into account a set of controls  $A_0$ , a dynamics  $f_0 : \Gamma \times A_0 \mapsto \mathbb{R}e_0$  and a running cost  $\ell_0 : \Gamma \times A_0 \mapsto \mathbb{R}$ . The assumptions made on  $A_0$ ,  $f_0$  and  $\ell_0$  are the following.

- (i)  $A_0$  is a non empty compact subset of the metric space  $A$ , disjoint from the other sets  $A_i$ ,  $i \in \{1, \dots, N\}$ .
- (ii) The function  $f_0$  satisfies the same boundedness and regularity properties as the functions  $f_i$ ,  $i \in \{1, \dots, N\}$ , described in [H0].
- (ii) The function  $\ell_0$  satisfies the same boundedness and regularity properties as the functions  $\ell_i$ ,  $i \in \{1, \dots, N\}$ , described in [H1].

We define

$$M = \{(x, a) : x \in \mathcal{S}, a \in A_i \text{ if } x \in \mathcal{S} \setminus \Gamma, \text{ and } a \in \cup_{i=0}^N A_i \text{ if } x \in \Gamma\},$$

the dynamics

$$\forall (x, a) \in M, \quad f(x, a) = \begin{cases} f_i(x, a) & \text{if } x \in \mathcal{P}_i \setminus \Gamma, i \in \{1, \dots, N\} \\ f_i(x, a) & \text{if } x \in \Gamma \text{ and } a \in A_i, i \in \{0, 1, \dots, N\}, \end{cases}$$

and the running cost

$$\forall (x, a) \in M, \quad \ell(x, a) = \begin{cases} \ell_i(x, a) & \text{if } x \in \mathcal{P}_i \setminus \Gamma, i \in \{1, \dots, N\} \\ \ell_i(x, a) & \text{if } x \in \Gamma \text{ and } a \in A_i, i \in \{0, 1, \dots, N\}. \end{cases}$$

The infinite horizon optimal control problem is then given by (2.7) and (2.10). Then, we consider the Hamilton-Jacobi equation (2.15) with the new definition of  $\text{FL}(x)$  :

$$\text{FL}(x) = \begin{cases} \text{FL}_i(x) & \text{if } x \text{ belongs to } \mathcal{P}_i \setminus \Gamma \\ \text{FL}_0(x) \cup \bigcup_{i=1, \dots, N} \text{FL}_i^+(x) & \text{if } x \in \Gamma, \end{cases}$$

where for  $x \in \Gamma$ ,  $\text{FL}_0(x) = \{(f_0(x, a), \ell_0(x, a)) : a \in A_0\}$ . The notion of viscosity sub and supersolutions of (2.15) can be also defined as in (2.16) and (2.17). We obtain that the value function is discontinuous viscosity solution of (2.15) in  $\mathcal{S}$  in the same manner as above, by passing by the relaxed Hamilton-Jacobi equation (2.28). Note that the key result to pass from (2.15) to (2.28), Lemma 2.2, in the present framework becomes

$$\begin{aligned} \tilde{f}\ell(x) &= \text{FL}(x) & \text{if } x \in \mathcal{S} \setminus \Gamma, \\ \tilde{f}\ell(x) &= \bigcup_{i=1, \dots, N} \overline{\text{co}} \left\{ \text{FL}_0(x) \cup \text{FL}_i^+(x) \cup \bigcup_{j \neq i} \left( \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R}) \right) \right\} & \text{if } x \in \Gamma. \end{aligned}$$

The proof of this can be made in the same way as above.

As previously, we have some equivalent definitions for (2.16) and (2.17) given by :

- An upper semi-continuous function  $u : \mathcal{S} \rightarrow \mathbb{R}$  is a subsolution of (2.15) in  $\mathcal{S}$  if for any  $x \in \mathcal{S}$ , any  $\varphi \in \mathcal{R}(\mathcal{S})$  s.t.  $u - \varphi$  has a local maximum point at  $x$ , then

$$\begin{aligned} \lambda u(x) + H_i(x, D(\varphi|_{\mathcal{P}_i})(x)) &\leq 0 & \text{if } x \in \mathcal{P}_i \setminus \Gamma, \\ \lambda u(x) + H_\Gamma(x, D(\varphi|_\Gamma)(x), D(\varphi|_{\mathcal{P}_1})(x), \dots, D(\varphi|_{\mathcal{P}_N})(x)) &\leq 0 & \text{if } x \in \Gamma. \end{aligned}$$

- A lower semi-continuous function  $u : \mathcal{S} \rightarrow \mathbb{R}$  is a supersolution of (2.15) if for any  $x \in \mathcal{S}$ , any  $\varphi \in \mathcal{R}(\mathcal{S})$  s.t.  $u - \varphi$  has a local minimum point at  $x$ , then

$$\begin{aligned} \lambda u(x) + H_i(x, D(\varphi|_{\mathcal{P}_i})(x)) &\geq 0 & \text{if } x \in \mathcal{P}_i \setminus \Gamma, \\ \lambda u(x) + H_\Gamma(x, D(\varphi|_\Gamma)(x), D(\varphi|_{\mathcal{P}_1})(x), \dots, D(\varphi|_{\mathcal{P}_N})(x)) &\geq 0 & \text{if } x \in \Gamma. \end{aligned}$$

Where the new definition of  $H_\Gamma : \Gamma \times \mathbb{R}e_0 \times \left(\prod_{i=1,\dots,N}(\mathbb{R}e_0 \times \mathbb{R}e_i)\right) \rightarrow \mathbb{R}$  is given by

$$H_\Gamma(x, p_0, p_1, \dots, p_N) = \max \left\{ H_0(x, p_0), \max_{i=1,\dots,N} H_i^+(x, p_i) \right\},$$

where the Hamiltonians  $H_i^+$  are defined in (2.20) and the Hamiltonian  $H_0 : \Gamma \times \mathbb{R}e_0 \mapsto \mathbb{R}$ , is defined by

$$H_0(x, p_0) = \max_{a \in A_0} (-f_0(x, a) \cdot p_0 - \ell_0(x, a)).$$

The tangential Hamiltonian at  $\Gamma$ ,  $H_\Gamma^T : \Gamma \times \mathbb{R}e_0 \mapsto \mathbb{R}$  is also slightly changed,

$$H_\Gamma^T(x, p) = \max \left\{ H_0(x, p), \max_{i=1,\dots,N} H_{\Gamma,i}^T(x, p) \right\}, \quad (4.1)$$

where the Hamiltonians  $H_{\Gamma,i}^T$  are defined in (2.22).

With these new definitions, all the results proved in § 3 hold with obvious modifications of the proofs. In particular,

- a subsolution of the present problem is also a subsolution of the former problem. So Lemma 2.7 (with remark 2.8) and equation (2.43) in Lemma 2.8 hold.
- Lemma 2.1 holds since it only involves the Hamiltonians  $H_i$ ,  $H_i^+$  and  $H_{\Gamma,i}^T$ .
- The proof of Theorem 2.4 can still be used. In particular, with the choice of  $(q_i)_{i=1,\dots,N}$  made in this proof, we have the identity

$$H_\Gamma(\bar{x}, D(\phi|_\Gamma)(\bar{x}) + q_1 e_1, \dots, D(\phi|_{\mathcal{P}_N})(\bar{x}) + q_N e_N) = H_\Gamma^T(\bar{x}, D(\phi|_\Gamma)(\bar{x})).$$

- The proofs of the regularisation results, Lemma 3.10 and Lemma 3.11, are unchanged.
- The proofs of the Comparison principles, Theorem 3.3 and Theorem 3.4, are unchanged.

## A Proof of Lemma 2.2

*Proof.* The proof of the equality  $\tilde{f}\ell(x) = \text{FL}(x)$  for  $x \in \mathcal{S} \setminus \Gamma$  is standard (see [4], Lemma 2.41, page 129), and the inclusions  $\overline{\text{co}} \left\{ \text{FL}_i^+(x) \cup \bigcup_{j \neq i} (\text{FL}_j(x) \cap (\Gamma \times \mathbb{R})) \right\} \subset \tilde{f}\ell(x)$  for  $x \in \Gamma$  and  $i \in \{1, \dots, N\}$  are proved by explicitly constructing trajectories, see [1]. We skip this part. This leads to

$$\begin{aligned} \text{FL}(x) &= \tilde{f}\ell(x) \quad \text{if } x \in \mathcal{S} \setminus \Gamma, \\ \bigcup_{i=1,\dots,N} \overline{\text{co}} \left\{ \text{FL}_i^+(x) \cup \bigcup_{j \neq i} (\text{FL}_j(x) \cap (\Gamma \times \mathbb{R})) \right\} &\subset \tilde{f}\ell(x) \quad \text{if } x \in \Gamma. \end{aligned}$$

We now prove the reverse inclusion. Let  $x \in \Gamma$ . For any  $(\zeta, \mu) \in \tilde{f}\ell(x)$ , there exists a sequence of admissible trajectories  $(y_n, \alpha_n) \in \mathcal{T}_x$  and a sequence of times  $t_n \rightarrow 0^+$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{t_n} \int_0^{t_n} f(y_n(t), \alpha_n(t)) dt = \zeta, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{t_n} \int_0^{t_n} \ell(y_n(t), \alpha_n(t)) dt = \mu.$$

First, remark that by construction  $\zeta$  necessarily belongs to  $\mathbb{R}e_0 \times \mathbb{R}^+ e_i$ , for some  $i \in \{1, \dots, N\}$ .

- If  $\zeta \notin \mathbb{R}e_0$ , then there exists an index  $i$  in  $\{1, \dots, N\}$  such that  $\zeta \in (\mathbb{R}e_0 \times \mathbb{R}^+ e_i) \setminus \mathbb{R}e_0$ : in this case,  $y_n(t_n) \in \mathcal{P}_i \setminus \Gamma$ . Hence,

$$y_n(t_n) = x + \sum_{j=1}^N \int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} dt + \int_0^{t_n} f(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \Gamma} dt, \quad (\text{A.1})$$

with

$$\begin{aligned} \int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} e_j dt &= 0 & \text{if } j \neq i, \\ \int_0^{t_n} f_i(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_i \setminus \Gamma} e_i dt &= y_n(t_n) \cdot e_i. \end{aligned} \quad (\text{A.2})$$

These identities are a consequence of Stampacchia's theorem: consider for example  $j \in \{1, \dots, N\}$  and the function  $\kappa_j : y \mapsto y 1_{y \in \mathcal{P}_j \setminus \Gamma} e_j$ . It is easy to check that  $t \mapsto \kappa_j(y_n(t))$  belongs to  $W^{1,\infty}(0, t_n)$  and that its weak derivative coincides almost everywhere with  $t \mapsto f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} e_j$ . This implies (A.2).

For  $j = 1, \dots, N$ , let  $t_{j,n}$  be defined by

$$t_{j,n} = \left| \left\{ t \in [0, t_n] : y_n(t) \in \mathcal{P}_j \setminus \Gamma \right\} \right|.$$

If  $j \neq i$  and  $t_{j,n} > 0$  then

$$\begin{aligned} & \frac{1}{t_{j,n}} \left( \int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} dt, \int_0^{t_n} \ell_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} dt \right) \\ &= \frac{1}{t_{j,n}} \left( \int_0^{t_n} f_j(x, \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} dt, \int_0^{t_n} \ell_j(x, \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} dt \right) + o(1) \end{aligned}$$

where  $o(1)$  is a vector tending to 0 as  $n \rightarrow \infty$ . Therefore, the distance of

$\frac{1}{t_{j,n}} \left( \int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} dt, \int_0^{t_n} \ell_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} dt \right)$  to the set  $\text{FL}_j(x)$

tends to 0. Moreover, according to (A.2), we have that  $\int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} e_j dt =$

0. Hence, the distance of  $\frac{1}{t_{j,n}} \left( e_j \int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} dt, \int_0^{t_n} \ell_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} dt \right)$

to the set  $(\text{FL}_j(x) \cap (\mathbb{R} e_0 \times \mathbb{R}))$  tends to zero as  $n$  tends to  $\infty$ .

If the set  $\{t : y_n(t) \in \Gamma\}$  has a nonzero measure, then

$$\begin{aligned} & \frac{1}{|\{t : y_n(t) \in \Gamma\}|} \left( \int_0^{t_n} f(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \Gamma} dt, \int_0^{t_n} \ell(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \Gamma} dt \right) \\ &= \frac{1}{|\{t : y_n(t) \in \Gamma\}|} \left( \int_0^{t_n} f(x, \alpha_n(t)) 1_{y_n(t) \in \Gamma} dt, \int_0^{t_n} \ell(x, \alpha_n(t)) 1_{y_n(t) \in \Gamma} dt \right) + o(1) \end{aligned}$$

Therefore, the distance of  $\frac{1}{|\{t : y_n(t) \in \Gamma\}|} \left( \int_0^{t_n} f(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \Gamma} dt, \int_0^{t_n} \ell(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \Gamma} dt \right)$

to the set  $\overline{\text{co}} \left\{ \bigcup_{j=1}^N \text{FL}_j(x) \right\}$  tends to zero as  $n$  tends to  $\infty$ . Moreover, from theorem 2.2,

$f(y_n(t), \alpha_n(t)) \in \mathbb{R} e_0$  almost everywhere on  $\{t : y_n(t) \in \Gamma\}$ . Therefore, the distance of

$\frac{1}{|\{t : y_n(t) \in \Gamma\}|} \left( \int_0^{t_n} f(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \Gamma} dt, \int_0^{t_n} \ell(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \Gamma} dt \right)$  to the set

$\overline{\text{co}} \left\{ \bigcup_{j=1}^N (\text{FL}_j(x) \cap (\mathbb{R} e_0 \times \mathbb{R})) \right\}$  tends to zero as  $n$  tends to  $\infty$ .

Finally, we know that  $T_{i,n} > 0$ .

$$\begin{aligned} & \frac{1}{t_{i,n}} \left( \int_0^{t_n} f_i(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_i \setminus \Gamma} dt, \int_0^{t_n} \ell_i(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_i \setminus \Gamma} dt \right) \\ &= \frac{1}{t_{i,n}} \left( \int_0^{t_n} f_i(x, \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_i \setminus \Gamma} dt, \int_0^{t_n} \ell_i(x, \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_i \setminus \Gamma} dt \right) + o(1) \end{aligned}$$

so the distance of

$\frac{1}{t_{i,n}} \left( \int_0^{t_n} f_i(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_i \setminus \Gamma} dt, \int_0^{t_n} \ell_i(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_i \setminus \Gamma} dt \right)$  to the set  $\text{FL}_i^+(x)$

tends to zero as  $n$  tends to  $\infty$ .

Combining all the observations above, we see that the distance of

$\left( \frac{1}{t_n} \int_0^{t_n} f(y_n(t), \alpha_n(t)) dt, \frac{1}{t_n} \int_0^{t_n} \ell(y_n(t), \alpha_n(t)) dt \right)$  to  $\overline{\text{co}} \left\{ \text{FL}_i^+(x) \cup \bigcup_{j \neq i} (\text{FL}_j(x) \cap (\mathbb{R} e_0 \times \mathbb{R})) \right\}$

tends to 0 as  $n \rightarrow \infty$ . Therefore  $(\zeta, \mu) \in \overline{\text{co}} \left\{ \text{FL}_i^+(x) \cup \bigcup_{j \neq i} (\text{FL}_j(x) \cap (\mathbb{R} e_0 \times \mathbb{R})) \right\}$ .

- If  $\zeta \in \mathbb{R}e_0$ , either there exists  $i$  such that  $y_n(t_n) \in \mathcal{P}_i \setminus \Gamma$  or  $y_n(t_n) \in \Gamma$ :
  - If  $y_n(t_n) \in \mathcal{P}_i \setminus \Gamma$ , then we can make exactly the same argument as above and conclude that  $(\zeta, \mu) \in \overline{\text{co}} \left\{ \text{FL}_i^+(x) \cup \bigcup_{j \neq i} \left( \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R}) \right) \right\}$ . Since  $\zeta \in \mathbb{R}e_0$ , we have in fact that  $(\zeta, \mu) \in \overline{\text{co}} \bigcup_{j=1}^N \left( \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R}) \right)$ .

- if  $y_n(t_n) \in \Gamma$ , according to Stampacchia theorem, we have that

$$\int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in \mathcal{P}_j \setminus \Gamma} e_j dt = 0 \text{ for all } j = 1, \dots, N. \text{ We can repeat the argument above, and obtain that } (\zeta, \mu) \in \overline{\text{co}} \left\{ \bigcup_{j=1}^N \left( \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R}) \right) \right\}.$$

□

## B Proof of Theorem 3.2

*Proof.* First, remark that in this proof, the notation  $o_\varepsilon(1)$  will denote an application independent of  $t$ , which tends to 0 as  $\varepsilon$  tends to 0 and that for  $k \in \mathbb{N}^*$  the notation  $O(t^k)$  will denote an application independent of  $\varepsilon$ , such that  $\frac{O(t^k)}{t^k}$  remains bounded as  $t$  tends to 0. **Show that  $v_\star$  is a supersolution of (2.15) :** for any  $x \in \mathcal{S}$ , let  $(x_\varepsilon)_{\varepsilon > 0}$  be a sequence such that  $x_\varepsilon$  tends to  $x$  when  $\varepsilon$  tends to 0 and  $v(x_\varepsilon)$  tends to  $v_\star(x)$  when  $\varepsilon$  tends to 0. Let  $\varphi$  be in  $\mathcal{R}(\mathcal{S})$  such that  $v_\star - \varphi$  has a local minimum at  $x$ , i.e. there exists  $r > 0$  such that

$$\forall y \in B(x, r) \cap \mathcal{S}, \quad v_\star(x) - \varphi(x) \leq v_\star(y) - \varphi(y). \quad (\text{B.1})$$

From the dynamic programming principle (Proposition 2.1), for any  $\varepsilon > 0$  and  $t > 0$ , there exists  $(\bar{y}_{\varepsilon,t}, \bar{\alpha}_{\varepsilon,t}) \in \mathcal{T}_{x_\varepsilon}$  such that

$$\begin{aligned} v(x_\varepsilon) &\geq \int_0^t \ell(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) e^{-\lambda s} ds + e^{-\lambda t} v(\bar{y}_{\varepsilon,t}(t)) - \varepsilon \\ &\geq \int_0^t \ell(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) e^{-\lambda s} ds + e^{-\lambda t} v_\star(\bar{y}_{\varepsilon,t}(t)) - \varepsilon. \end{aligned}$$

Then, according to (B.1), for  $\varepsilon$  and  $t > 0$  small enough we have

$$v(x_\varepsilon) - v_\star(x) \geq \int_0^t \ell(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) e^{-\lambda s} ds + v_\star(x)(e^{-\lambda t} - 1) + (\varphi(\bar{y}_{\varepsilon,t}(t)) - \varphi(x)) e^{-\lambda t} - \varepsilon.$$

Using that  $v(x_\varepsilon) - v_\star(x) = o_\varepsilon(1)$ , that  $\int_0^t \ell(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) e^{-\lambda s} ds = \int_0^t \ell(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds + O(t^2)$  and that  $(\varphi(\bar{y}_{\varepsilon,t}(t)) - \varphi(x)) e^{-\lambda t} = (\varphi(\bar{y}_{\varepsilon,t}(t)) - \varphi(x)) + t o_\varepsilon(1) + O(t^2)$ , we finally obtain

$$0 \geq \int_0^t \ell(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds + v_\star(x)(e^{-\lambda t} - 1) + \varphi(\bar{y}_{\varepsilon,t}(t)) - \varphi(x) + t o_\varepsilon(1) + O(t^2) + o_\varepsilon(1). \quad (\text{B.2})$$

- **If  $x \in \mathcal{P}_i \setminus \Gamma$  :** since  $x_\varepsilon$  tends to  $x$  belonging to  $\mathcal{P}_i \setminus \Gamma$  as  $\varepsilon$  tends to 0 and since the dynamic  $f$  is bounded, see remark 2.2, there exists  $\bar{t} > 0$  such that for any  $t \in (0, \bar{t})$ ,  $\bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_i \setminus \Gamma$  for any  $s \in (0, t)$ . Then, the inequality (B.2) can be rewritten as follows

$$\begin{aligned} 0 &\geq \int_0^t \ell_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) + D(\varphi|_{\mathcal{P}_i})(\bar{y}_{\varepsilon,t}(s)) \cdot f_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds + v_\star(x)(e^{-\lambda t} - 1) \\ &\quad + \varphi(x_\varepsilon) - \varphi(x) + t o_\varepsilon(1) + O(t^2) + o_\varepsilon(1). \end{aligned}$$

Using that  $\varphi(x_\varepsilon) - \varphi(x) = o_\varepsilon(1)$  and that  $D(\varphi|_{\mathcal{P}_i})(\bar{y}_{\varepsilon,t}(t)) = D(\varphi|_{\mathcal{P}_i})(x) + o_\varepsilon(1) + O(t)$ , we get that

$$\begin{aligned} 0 &\geq \int_0^t \ell_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) + D(\varphi|_{\mathcal{P}_i})(x) \cdot f_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds + v_\star(x)(e^{-\lambda t} - 1) \\ &\quad + t o_\varepsilon(1) + O(t^2) + o_\varepsilon(1). \end{aligned} \quad (\text{B.3})$$



It is easy to check that  $\frac{1}{t} \left( \int_0^t f_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds, \int_0^t \ell_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds \right)$  is at a distance to  $\text{FL}_i(x)$  of the order of  $o_\varepsilon(1) + O(t)$ . Thus,

$$0 \geq v_\star(x)(e^{-\lambda t} - 1) - t \left( \max_{(\xi, \zeta) \in \text{FL}_i(x)} \{-D(\varphi|_{\mathcal{P}_i})(x, \xi) - \zeta\} \right) + to_\varepsilon(1) + O(t^2) + o_\varepsilon(1).$$

Finally, dividing the latter inequality by  $t$ , taking the limit as  $\varepsilon$  tends to 0 and in a second time the limit as  $t$  tends to 0 we get the desired inequality

$$\lambda v_\star(x) + \max_{(\xi, \zeta) \in \text{FL}_i(x)} \{-D(\varphi|_{\mathcal{P}_i})(x, \xi) - \zeta\} \geq 0.$$

- **If  $x \in \Gamma$  and  $x_\varepsilon \in \mathcal{P}_i \setminus \Gamma$  :** Let  $\tau_{\varepsilon,t} > 0$  be the exit time of  $\bar{y}_{\varepsilon,t}$  from  $\mathcal{P}_i \setminus \Gamma$ . Up to the extraction of a subsequence, we may assume that either  $\tau_{\varepsilon,t} \leq t$  for all  $\varepsilon > 0$  or that  $\tau_{\varepsilon,t} > t$  for all  $\varepsilon$ .

**If  $\tau_{\varepsilon,t} > t$  :** The same calculations as in the case where  $x \in \mathcal{P}_i \setminus \Gamma$  give us (B.3). As above  $\frac{1}{t} \left( \int_0^t f_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds, \int_0^t \ell_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds \right)$  is at a distance to  $\text{FL}_i(x)$  of the order of  $o_\varepsilon(1) + O(t)$ . But this time, we have that  $\int_0^t f_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds \cdot e_i \geq -x_\varepsilon \cdot e_i = o_\varepsilon(1)$  for all  $t, \varepsilon > 0$  and then we have the more specific information that  $\frac{1}{t} \left( \int_0^t f_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds, \int_0^t \ell_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds \right)$  is at a distance to  $\text{FL}_i^+(x)$  of the order of  $\frac{o_\varepsilon(1)}{t} + o_\varepsilon(1) + O(t)$ . Finally, (B.3) give us as desired

$$\lambda v_\star(x) + \max_{i \in \{1, \dots, N\}} \max_{(\xi, \zeta) \in \text{FL}_i^+(x)} \{-D(\varphi|_{\mathcal{P}_i})(x, \xi) - \zeta\} \geq 0.$$

**If  $\tau_{\varepsilon,t} \leq t$  :** Then, the inequality (B.2) can be written as follow

$$\begin{aligned} 0 \geq & \int_0^{\tau_{\varepsilon,t}} \ell_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) + D(\varphi|_{\mathcal{P}_i})(\bar{y}_{\varepsilon,t}(s)) \cdot f_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds \\ & + \sum_{j=1}^N \int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_j \setminus \Gamma\}} [\ell_j(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) + D(\varphi|_{\mathcal{P}_j})(\bar{y}_{\varepsilon,t}(s)) \cdot f_j(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s))] ds \\ & + \int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \Gamma\}} [\ell(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) + D(\varphi|_{\Gamma})(\bar{y}_{\varepsilon,t}(s)) \cdot f(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s))] ds \\ & + v_\star(x)(e^{-\lambda t} - 1) + \varphi(x_\varepsilon) - \varphi(x) + to_\varepsilon(1) + O(t^2) + o_\varepsilon(1). \end{aligned} \tag{B.4}$$

Note that to obtain the third line of this inequality, we use Point 3 of Theorem 3.1. To obtain the supersolution inequality, we have to deal with each term of the inequality (B.4) individually.

→ Let  $j \in \{1, \dots, N\}$  be such that  $\bar{y}_{\varepsilon,t}(t) \notin \mathcal{P}_j \setminus \Gamma$ . Then  $\int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_j \setminus \Gamma\}} f_j(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) \cdot e_j ds = 0$  and

$\frac{1}{|\{s : \bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_j \setminus \Gamma\}|} \left( \int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_j \setminus \Gamma\}} f_j(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds, \int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_j \setminus \Gamma\}} \ell_j(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds \right)$  is at a distance to  $\text{FL}_i(x) \cap (\mathbb{R}e_0 \times \mathbb{R})$  of the order of  $o_\varepsilon(1) + O(t)$ . Thus, we get

$$\begin{aligned} & \int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_j \setminus \Gamma\}} [\ell_j(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) + D(\varphi|_{\mathcal{P}_j})(\bar{y}_{\varepsilon,t}(s)) \cdot f_j(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s))] ds \\ & \geq |\{s : \bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_j \setminus \Gamma\}| \left( -\max_{(\xi, \zeta) \in \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R})} \{-D\varphi(x, \xi) - \zeta\} + o_\varepsilon(1) + O(t) \right). \end{aligned} \tag{B.5}$$

→ If there exists one  $k \in \{1, \dots, N\}$ , such that  $\bar{y}_{\varepsilon,t}(t) \in \mathcal{P}_k \setminus \Gamma$ . In this case, we have that  $\int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_k \setminus \Gamma\}} f_k(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) \cdot e_k ds = \bar{y}_{\varepsilon,t}(t) \cdot e_k > 0$  and then that

$\frac{1}{|\{s : \bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_k \setminus \Gamma\}|} \left( \int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_k \setminus \Gamma\}} f_k(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds, \int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_k \setminus \Gamma\}} \ell_k(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds \right)$   
is at a distance to  $\text{FL}_k^+(x)$  of the order of  $o_\varepsilon(1) + O(t)$ . Therefore, we get that

$$\begin{aligned} & \int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_k \setminus \Gamma\}} [\ell_k(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) + D(\varphi|_{\mathcal{P}_k})(\bar{y}_{\varepsilon,t}(s)) \cdot f_k(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s))] ds \\ & \geq |\{s : \bar{y}_{\varepsilon,t}(s) \in \mathcal{P}_k \setminus \Gamma\}| \left( -\max_{(\xi, \zeta) \in \text{FL}_k^+(x)} \{-D\varphi(x, \xi) - \zeta\} + o_\varepsilon(1) + O(t) \right). \end{aligned} \quad (\text{B.6})$$

→ From Point 3 of Theorem 3.1, we know that  $f(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) \in \mathbb{R}e_0$  almost everywhere on  $\{s : \bar{y}_{\varepsilon,t}(s) \in \Gamma\}$ . Therefore,

$$\frac{1}{|\{s : \bar{y}_{\varepsilon,t}(s) \in \Gamma\}|} \left( \int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \Gamma\}} f(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds, \int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \Gamma\}} \ell(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds \right)$$

is at a distance to  $\overline{\text{co}} \left\{ \bigcup_{j=1}^N \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R}) \right\}$  of the order of  $o_\varepsilon(1) + O(t)$  and then

$$\begin{aligned} & \int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \Gamma\}} [\ell(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) + D_\Gamma \varphi(\bar{y}_{\varepsilon,t}(s)) \cdot f(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s))] ds \\ & \geq |\{s : \bar{y}_{\varepsilon,t}(s) \in \Gamma\}| \left( -\max_{(\xi, \zeta) \in \overline{\text{co}} \left\{ \bigcup_{j=1}^N \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R}) \right\}} \{-D\varphi(x, \xi) - \zeta\} + o_\varepsilon(1) + O(t) \right). \end{aligned} \quad (\text{B.7})$$

However, from the piecewise linearity of the function  $(\xi, \zeta) \mapsto -D\varphi(x, \xi) - \zeta$  we have that

$$\max_{(\xi, \zeta) \in \overline{\text{co}} \left\{ \bigcup_{j=1}^N \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R}) \right\}} \{-D\varphi(x, \xi) - \zeta\} = \max_{j \in \{1, \dots, N\}} \left\{ \max_{(\xi, \zeta) \in \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R})} \{-D\varphi(x, \xi) - \zeta\} \right\}.$$

Therefore, (B.7) give us finally

$$\begin{aligned} & \int_{\tau_{\varepsilon,t}}^t 1_{\{\bar{y}_{\varepsilon,t}(s) \in \Gamma\}} [\ell(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) + D_\Gamma \varphi(\bar{y}_{\varepsilon,t}(s)) \cdot f(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s))] ds \\ & \geq |\{s : \bar{y}_{\varepsilon,t}(s) \in \Gamma\}| \left( -\max_{j \in \{1, \dots, N\}} \left\{ \max_{(\xi, \zeta) \in \text{FL}_j(x) \cap (\mathbb{R}e_0 \times \mathbb{R})} \{-D\varphi(x, \xi) - \zeta\} \right\} + o_\varepsilon(1) + O(t) \right). \end{aligned} \quad (\text{B.8})$$

→ It remains to deal with  $\int_0^{\tau_{\varepsilon,t}} \ell_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) + D(\varphi|_{\mathcal{P}_i})(\bar{y}_{\varepsilon,t}(s)) \cdot f_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds$ .

It is the term the most tricky one because  $\int_0^{\tau_{\varepsilon,t}} f_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) \cdot e_i ds = -x_\varepsilon \cdot e_i < 0$  generates some outgoing directions. To conclude, we have to use that  $x_\varepsilon \cdot e_i = o_\varepsilon(1)$ . As a consequence, up to the extraction of a subsequence, we may assume that either  $\lim_{\varepsilon \rightarrow 0} \frac{|x_\varepsilon \cdot e_i|}{\tau_{\varepsilon,t}} = 0$  or  $C_1 \leq \frac{|x_\varepsilon \cdot e_i|}{\tau_{\varepsilon,t}} \leq C_2$ , for all  $\varepsilon$ , for some positive constants  $C_1, C_2$ .

If  $\lim_{\varepsilon \rightarrow 0} \frac{|x_\varepsilon \cdot e_i|}{\tau_{\varepsilon,t}} = 0$ , it is simple to see that  $\frac{1}{\tau_{\varepsilon,t}} \left( \int_0^{\tau_{\varepsilon,t}} f_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds, \int_0^{\tau_{\varepsilon,t}} \ell_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds \right)$   
is at a distance to  $\text{FL}_i(x) \cap (\mathbb{R}e_0 \times \mathbb{R})$  of the order of  $o_\varepsilon(1) + O(t)$ . Then, we get

$$\begin{aligned} & \int_0^{\tau_{\varepsilon,t}} \ell_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) + D(\varphi|_{\mathcal{P}_i})(\bar{y}_{\varepsilon,t}(s)) \cdot f_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds \\ & \geq \tau_{\varepsilon,t} \left( -\max_{(\xi, \zeta) \in \text{FL}_i(x) \cap (\mathbb{R}e_0 \times \mathbb{R})} \{-D\varphi(x, \xi) - \zeta\} + o_\varepsilon(1) + O(t) \right). \end{aligned} \quad (\text{B.9})$$

If  $C_1 \leq \frac{|x_\varepsilon \cdot e_i|}{\tau_{\varepsilon,t}} \leq C_2$ , we can directly prove that

$$\int_0^{\tau_{\varepsilon,t}} \ell_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) + D(\varphi|_{\mathcal{P}_i})(\bar{y}_{\varepsilon,t}(s)) \cdot f_i(\bar{y}_{\varepsilon,t}(s), \bar{\alpha}_{\varepsilon,t}(s)) ds = O(x_\varepsilon \cdot e_i) = o_\varepsilon(1). \quad (\text{B.10})$$

Finally, if we put together all the informations of (B.5), (B.6), (B.8), (B.9), (B.10) and the fact that  $\varphi(x_\varepsilon) - \varphi(x) = o_\varepsilon(1)$ , the inequality (B.2) gives us

$$0 \geq -t \max_{j \in \{1, \dots, N\}} \left\{ \max_{(\xi, \zeta) \in \text{FL}_j^+(x)} \{-D(\varphi|_{\mathcal{P}_j})(x) \cdot \xi - \zeta\} + v_\star(x)(e^{-\lambda t} - 1) + t o_\varepsilon(1) + O(t^2) + o_\varepsilon(1) \right\}.$$

Dividing this last inequality by  $t$ , taking the limit as  $\varepsilon$  tends to 0 and in a second time the limit as  $t$  tends to 0 we get the wanted inequality

$$\lambda v_\star(x) + \max_{(\xi, \zeta) \in \text{FL}(x)} \{-D\varphi(x, \xi) - \zeta\} \geq 0.$$

- **If  $x \in \Gamma$  and  $x_\varepsilon \in \Gamma$  :** In this case, the inequality (B.2) can be written as follow

$$\begin{aligned} 0 \geq & + \sum_{j=1}^N \int_0^t 1_{\{\bar{y}_{\varepsilon, t}(s) \in \mathcal{P}_j \setminus \Gamma\}} [\ell_j(\bar{y}_{\varepsilon, t}(s), \bar{\alpha}_{\varepsilon, t}(s)) + D(\varphi|_{\mathcal{P}_j})(\bar{y}_{\varepsilon, t}(s)) \cdot f_j(\bar{y}_{\varepsilon, t}(s), \bar{\alpha}_{\varepsilon, t}(s))] ds \\ & + \int_0^t 1_{\{\bar{y}_{\varepsilon, t}(s) \in \Gamma\}} [\ell(\bar{y}_{\varepsilon, t}(s), \bar{\alpha}_{\varepsilon, t}(s)) + D_\Gamma \varphi(\bar{y}_{\varepsilon, t}(s)) \cdot f(\bar{y}_{\varepsilon, t}(s), \bar{\alpha}_{\varepsilon, t}(s))] ds \\ & + v_\star(x)(e^{-\lambda t} - 1) + \varphi(x_\varepsilon) - \varphi(x) + t o_\varepsilon(1) + O(t^2) + o_\varepsilon(1). \end{aligned}$$

The same study term by term that in the previous case gives us the desired result.

**Show that  $v^\star$  is a subsolution of (2.15) :** for  $x \in \mathcal{S}$  let  $(x_\varepsilon)_{\varepsilon > 0}$  be a sequence such that  $x_\varepsilon$  tends to  $x$  as  $\varepsilon$  tends to 0 and  $v(x_\varepsilon)$  tends to  $v^\star(x)$  as  $\varepsilon$  tends to 0. Let  $\varphi$  be in  $\mathcal{R}(\mathcal{S})$  such that  $v^\star - \varphi$  has a local maximum at  $x$ , i.e. there exists  $r > 0$  such that

$$\forall y \in B(x, r) \cap \mathcal{S}, \quad v^\star(x) - \varphi(x) \geq v^\star(y) - \varphi(y). \quad (\text{B.11})$$

To prove that  $v^\star$  is a subsolution, according to Corollary 3.1, it is enough to show that for any  $k \in \{1, \dots, N\}$ ,

$$v^\star(x) + \sup_{a \in A_k \text{ s.t. } f_k(x, a) \cdot e_k > 0} (-D(\varphi|_{\mathcal{P}_k})(x) \cdot f_k(x, a) - \ell_k(x, a)) \leq 0. \quad (\text{B.12})$$

Let  $\bar{a} \in A_k$  be such that  $f_k(x, \bar{a}) \cdot e_k = \delta_{\bar{a}} > 0$ .

- **If  $x_\varepsilon \in \mathcal{P}_i \setminus \Gamma$  :** For all  $\varepsilon > 0$ , let  $(\bar{y}_\varepsilon, \bar{\alpha}_\varepsilon) \in \mathcal{T}_{x_\varepsilon}$  be an admissible controlled trajectory given by Lemma 3.2 and  $\bar{\tau}_\varepsilon (\leq C x_\varepsilon \cdot e_i)$  its exit time from  $\mathcal{P}_i \setminus \Gamma$ . So, we consider  $\tilde{y} : [0, \tilde{t}) \rightarrow \mathcal{P}_k$  the maximal solution of the integral equation  $\bar{y}(t) = \bar{y}_\varepsilon(\bar{\tau}_\varepsilon) + \int_0^t f_k(\bar{y}(s), \bar{a}) ds$ . According to the assumptions [H0] and  $\tilde{\text{H3}}$ , we can check that  $\tilde{t} \geq \frac{\delta_{\bar{a}}}{2L_f M_f}$ . Then, we introduce  $(y_\varepsilon, \alpha_\varepsilon)$  the admissible controlled trajectory of  $\mathcal{T}_{x_\varepsilon}$  defined in  $[0, \bar{\tau}_\varepsilon + \tilde{t})$  as follow

$$(y_\varepsilon(t), \alpha_\varepsilon(t)) = \begin{cases} (\bar{y}_\varepsilon(t), \bar{\alpha}_\varepsilon(t)) & \text{if } t < \bar{\tau}_\varepsilon, \\ (\bar{y}(t - \bar{\tau}_\varepsilon), \bar{a}) & \text{if } \bar{\tau}_\varepsilon \leq t < \bar{\tau}_\varepsilon + \tilde{t}. \end{cases}$$

Note that by construction, for  $t \in (\bar{\tau}_\varepsilon, \tilde{t})$  small enough, we have  $y_\varepsilon(s) \in \mathcal{P}_i \setminus \Gamma$  for  $s \in (0, \bar{\tau}_\varepsilon)$  and  $y_\varepsilon(s) \in \mathcal{P}_k \setminus \Gamma$  for  $s \in (\bar{\tau}_\varepsilon, t)$ . In the sequel we will consider such a  $t$ .

From the dynamic programming principle, Proposition 2.1, we have

$$\begin{aligned} v(x_\varepsilon) & \leq \int_0^t \ell(y_\varepsilon(s), \alpha_\varepsilon(s)) e^{-\lambda s} ds + e^{-\lambda t} v(y_\varepsilon(t)) \\ & \leq \int_0^t \ell(y_\varepsilon(s), \alpha_\varepsilon(s)) e^{-\lambda s} ds + e^{-\lambda t} v^\star(y_\varepsilon(t)). \end{aligned}$$

Then, according to (B.11), for  $\varepsilon$  small enough we have

$$v(x_\varepsilon) - v^\star(x) \leq \int_0^t \ell(y_\varepsilon(s), \alpha_\varepsilon(s)) e^{-\lambda s} ds + v^\star(x)(e^{-\lambda t} - 1) + (\varphi(y_\varepsilon(t)) - \varphi(x)) e^{-\lambda t}.$$

By similar arguments as above, we can deduce from this inequality the following

$$0 \leq \int_0^t \ell(y_{\varepsilon, t}(s), \alpha_{\varepsilon, t}(s)) ds + v^\star(x)(e^{-\lambda t} - 1) + \varphi(y_{\varepsilon, t}(t)) - \varphi(x) + t o_\varepsilon(1) + O(t^2) + o_\varepsilon(1).$$

And by construction of the admissible controlled trajectory  $(y_\varepsilon, \alpha_\varepsilon)$ , this inequality can be written as follows

$$\begin{aligned} 0 \leq & \int_0^{\bar{\tau}_\varepsilon} \ell_i(y_\varepsilon(s), \alpha_\varepsilon(s)) + D(\varphi|_{\mathcal{P}_i})(y_\varepsilon(s)) \cdot f_i(y_\varepsilon(s), \alpha_\varepsilon(s)) ds \\ & + \int_{\bar{\tau}_\varepsilon}^t \ell_k(y_\varepsilon(s), \bar{a}) + D(\varphi|_{\mathcal{P}_k})(y_\varepsilon(s)) \cdot f_k(y_\varepsilon(s), \bar{a}) ds \\ & + v^*(x)(e^{-\lambda t} - 1) + \varphi(x_\varepsilon) - \varphi(x) + t o_\varepsilon(1) + O(t^2) + o_\varepsilon(1). \end{aligned} \quad (\text{B.13})$$

So, using that  $\int_0^{\bar{\tau}_\varepsilon} \ell_i(y_\varepsilon(s), \alpha_\varepsilon(s)) + D(\varphi|_{\mathcal{P}_i})(y_\varepsilon(s)) \cdot f_i(y_\varepsilon(s), \alpha_\varepsilon(s)) ds = o_\varepsilon(1)$

and  $\frac{1}{t - \bar{\tau}_\varepsilon} \int_{\bar{\tau}_\varepsilon}^t \ell_k(y_\varepsilon(s), \bar{a}) + D(\varphi|_{\mathcal{P}_k})(y_\varepsilon(s)) \cdot f_k(y_\varepsilon(s), \bar{a}) ds$   
 $= D(\varphi|_{\mathcal{P}_k})(x) \cdot f_k(x, \bar{a}) + \ell_k(x, \bar{a}) + o_\varepsilon(1) + O(t)$ . Thus, the inequality (B.13) gives us

$$0 \geq v^*(x)(e^{-\lambda t} - 1) + t(D(\varphi|_{\mathcal{P}_k})(x) \cdot f_k(x, \bar{a}) + \ell_k(x, \bar{a})) + t o_\varepsilon(1) + O(t^2) + o_\varepsilon(1).$$

Dividing this last inequality by  $t$ , taking the limit as  $\varepsilon$  tends to 0 and in a second time the limit as  $t$  tends to 0 we give us

$$\lambda v^*(x) - D(\varphi|_{\mathcal{P}_k})(x) \cdot f_k(x, \bar{a}) - \ell_k(x, \bar{a}) \leq 0.$$

Since this inequality is true for any  $\bar{a} \in A_k$  such that  $f_k(x, \bar{a}) \cdot e_k > 0$ , we finally get (B.12).

- **If  $x_\varepsilon \in \Gamma$  :** Then, the same argument as above can be used. The only difference is that for the construction of the admissible controlled trajectory  $(y_\varepsilon, \alpha_\varepsilon)$  we do not need to join  $x_\varepsilon$  to  $\Gamma$ . Consequently, the calculations that follow are slightly simpler.

□

**Acknowledgement** The author was partially funded by the ANR project ANR-12-BS01-0008-01.

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